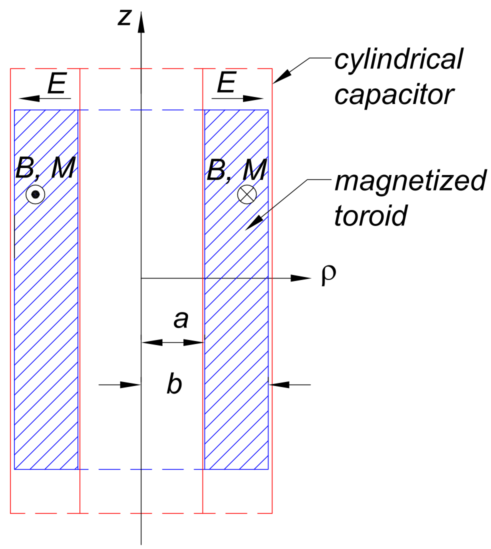
“Hidden” Momentum in a Magnetized Toroid

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(March 29, 2015; updated October 19, 2015)

1 Problem

Discuss the motion of a uniformly, azimuthally magnetized toroid that is inside a coaxial, cylindrical capacitor when the magnetization goes to zero.



The two electrodes of the capacitor, and the toroid, can move without friction along their common axis. Ignore gravity.

This problem, a variant of Cullwick's paradox [1], was suggested by Vladimir Hnizdo.

2 Solution

2.1 Electric and Magnetic Fields

The toroid has uniform azimuthal magnetization $\mathbf{M} = M\,\hat{\boldsymbol{\phi}}$ in cylindrical coordinates (ϱ, ϕ, z). The total magnetic moment is zero, $\mathbf{m}_{\rm total} = \int \mathbf{M}\, d{\rm Vol} = 0$.

For uniform magnetization $\mathbf{M}$, the volume density $\rho_m = -\boldsymbol{\nabla}\cdot\mathbf{M}$ of “fictitious” magnetic charges is zero.[1] For a toroid with azimuthal magnetization, the surface density $\sigma_m = \mathbf{M}\cdot\hat{\mathbf{n}}$ of “fictitious” magnetic charges is also zero (where $\hat{\mathbf{n}}$ is normal to the surface). Then, there are no sources for the $\mathbf{H}$-field, which is hence zero. That is, $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = 0$ everywhere, such that $\mathbf{B} = 4\pi\mathbf{M}$ is purely azimuthal inside the toroid and zero outside it.[2]

[1]For discussion of “fictitious” magnetic poles, see, for example, Appendix A of [2].

[2]*June 6, 2022.* For linear magnetic media, the magnetic field energy is $U_m = \int \mathbf{B}\cdot\mathbf{H}\, d{\rm Vol}/8\pi$, which is zero in the present example. However, this expression for the energy does not apply to a permanent magnet. Instead, one should consider $U_m = -\int \mathbf{M}\cdot\mathbf{B}\, d{\rm Vol} = -4\pi M^2\,{\rm Vol}$. For a misunderstanding of this issue, see Appendix 3 of [3].

We take the inner and outer radii of the toroid to be a and b, respectively, and suppose the toroid has length l in the axial (z) direction large compared to b. The conductors of the cylindrical capacitor are at radii a^- and b^+, but do not touch the toroid.

Inside the capacitor, away from its ends, the (static) electric field is radial and given by,

$$E_r(a^- < \varrho < b^+) = E_0 \frac{a}{\varrho}\,. \tag{1}$$

The surface charge densities on the inner and outer electrodes (away from their ends) are,

$$\sigma_{a^-} = \frac{E_r(\varrho = a^-)}{4\pi} = \frac{E_0}{4\pi}\,, \qquad \sigma_{b^+} = -\frac{E_r(\varrho = b^+)}{4\pi} = -E_0 \frac{a}{4\pi b}\,. \tag{2}$$

The azimuthal magnetic field $\mathbf{B}$, which is deducible from an vector potential $\mathbf{A}$ that is approximately azimuthal for z not close to the ends of the toroid, is given by,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \qquad B_\phi(|z| < l/2) = \begin{cases} 0 & (\varrho < a), \\ 4\pi M & (a < \varrho < b), \\ 0 & (\varrho > b). \end{cases} \approx -\frac{\partial A_z}{\partial \varrho} \tag{3}$$

We take the vector potential to be zero at $\varrho = \infty$, such that,

$$A_z(|z| < l/2) \approx 4\pi M \begin{cases} b - a & (\varrho < a), \\ b - \varrho & (a < \varrho < b), \\ 0 & (\varrho > b). \end{cases} \tag{4}$$

When the magnetization drops at rate $\dot{M} < 0$, the induced electric field is,

$$E_{\mathrm{ind},z}(|z| < l/2) = -\frac{1}{c}\frac{\partial A_z}{\partial t} \approx -\frac{4\pi \dot{M}}{c} \begin{cases} b - a & (\varrho < a), \\ b - \varrho & (a < \varrho < b), \\ 0 & (\varrho > b). \end{cases} \tag{5}$$

2.2 Motion of the Capacitor

As the magnetization drops to zero, $\dot{M} < 0$, there is no induced electric field at the outer conductor ($\varrho = b$), while the field on the inner conductor ($\varrho = a$) is in the $+z$ direction.. If the inner and outer conductors are free to move separately, only the inner conductor will move, and in the $+z$ direction as the surface charge density σ_a is positive. The force per unit length on the inner conductor is,

$$F_{\text{inner conductor},z}(\varrho = a^-) = 2\pi a \sigma_{a^-} E_{\mathrm{ind},z}(\varrho = a^-) = -\frac{2\pi E_0 \dot{M} a(b-a)}{c}\,, \tag{6}$$

so the inner conductor takes on mechanical momentum per unit length,

$$P_{\text{inner conductor},z} = \int F_z\, dt = \frac{2\pi E_0 M a(b-a)}{c}\,, \tag{7}$$

when the magnetization drops from M to zero. If the conductors are free to move, the inner conductor has final velocity in the $+z$ direction, while the outer conductor remains at rest.

2.3 Motion of the Toroid

The magnetized toroid is a collection of Ampèrian magnetic dipoles (with azimuthal magnetic moments $\mathbf{m}$), meaning that their fields are consistent with being generated by electrical currents (rather than by true (Gilbertian) magnetic charges, *i.e.*, magnetic monopoles). The force on an Ampèrian magnetic dipole $\mathbf{m}$ due to electromagnetic fields (generated by electric charges and currents, including Ampèrian magnetization) is given, for example, in eq. (18) of [4],

$$\mathbf{F}_m = (\mathbf{m} \cdot \boldsymbol{\nabla})\mathbf{B} + \mathbf{m} \times \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + c\mathbf{m} \times (\boldsymbol{\nabla} \times \mathbf{M}). \tag{8}$$

In the present example, $\mathbf{m}$, $\mathbf{B}$ and $\mathbf{M}$ are all azimuthal, so the force reduces to,

$$\mathbf{F}_m = \mathbf{m} \times \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\,, \tag{9}$$

which is in the radial direction, recalling eq. (5). The total force on all the magnetic dipoles in the (azimuthally symmetric) toroid is therefore zero.

This suggests that the toroid does not move as the magnetization falls to zero, while the inner conductor of the capacitor does move. If so, this system would violate conservation of momentum!

2.4 "Hidden" Mechanical Momentum

A related example, in which momentum appeared not to be conserved in an electromechanical system initially "at rest", led Shockley in 1967 to develop the notion of "hidden" mechanical momentum [5], *i.e.*, that the total mechanical momentum of a system is not necessarily the product of its mechanical mass/energy and the velocity of the center of mechanical mass/energy.[3,4] This notion was clarified in important ways by Coleman and Van Vleck [10], and then Furry [11] deduced that the "hidden" mechanical momentum of an Ampèrian magnetic dipole $\mathbf{m}$, that is at rest in an electric field due to electric charges also at rest, is,[5]

$$\mathbf{P}_{\text{hidden,mech}} = \mathbf{m} \times \frac{\mathbf{E}}{c} = -\mathbf{P}_{\text{EM}} = -\int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d\text{Vol}. \tag{10}$$

The view is that the total mechanical momentum of a magnetic dipole of mass m and velocity $\mathbf{v}$ consists of its "overt" momentum $\mathsf{m}\mathbf{v}$ plus its "hidden" momentum (10),

$$\mathbf{P}_{m,\text{mech}} = \mathbf{P}_{m,\text{overt}} + \mathbf{P}_{m,\text{hidden}} = \mathsf{m}\mathbf{v} + \mathbf{m} \times \frac{\mathbf{E}}{c}\,. \tag{11}$$

---

[3]The first such example was given in 1904 by J.J. Thomson on p. 348 of [6]. See also [7]. For examples with "hidden" mechanical momentum in systems with an electric dipole in a magnetic field due to current loops, all "at rest", such that various equal and opposite "overt" mechanical momenta arise as the electromagnetic fields are brought to zero in various ways, see [8], especially secs. IV and V.

[4]For a general discussion on the meaning of "hidden" momentum see [9].

[5]Furry also noted that a Gilbertian magnetic dipole (composed of opposite, true magnetic charges) has no "hidden" mechanical momentum when in a static electric field. That is, there must be moving electric charge for a system to posses "hidden" mechanical momentum, such that it is plausible to associate the momentum with moving charges. This theme is also sketched in footnote 9 of [10]. See also [7, 12].

Then, Newton's second law for the magnetic dipole is,[6]

$$\mathbf{F}_m = \frac{d\mathbf{P}_{m,\mathrm{mech}}}{dt} = \frac{d\mathbf{P}_{m,\mathrm{overt}}}{dt} + \frac{d\mathbf{m}}{dt} \times \frac{\mathbf{E}}{c} + \mathbf{m} \times \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \,. \tag{12}$$

Noting that the force on the dipole in the present example is given by eq. (9), we find that,

$$\frac{d\mathbf{P}_{m,\mathrm{overt}}}{dt} = -\frac{d\mathbf{m}}{dt} \times \frac{\mathbf{E}}{c} \,. \tag{13}$$

As the magnetic moment drops from $\mathbf{m}$ to zero, the "overt" mechanical momentum of the dipole rises to,

$$\mathbf{P}_{m,\mathrm{overt,final}} = \mathbf{m} \times \frac{\mathbf{E}}{c} = \mathbf{P}_{m,\mathrm{hidden,initial}} \,. \tag{14}$$

We interpret this as the initial "hidden" mechanical momentum of the magnetic dipole at rest in an electric field being converted into "overt" mechanical momentum, *i.e.*, motion of the dipole as a whole, when its magnetic moment disappears.

Summing over the magnetic dipoles in the toroid, we obtain Newton's second law for the densities of force, of "overt" mechanical momentum, and of magnetization,

$$\mathbf{f}_M = \mathbf{M} \times \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{d}{dt}\left(\mathbf{p}_{\mathrm{toroid,overt}} + \frac{\mathbf{M}\times\mathbf{E}}{c}\right), \qquad \frac{d\mathbf{p}_{\mathrm{toroid,overt}}}{dt} = -\frac{\partial \mathbf{M}}{\partial t} \times \frac{\mathbf{E}}{c} \,. \tag{15}$$

As the magnetization $\mathbf{M}$ drops to zero, the overt mechanical momentum of the toroid changes, until finally the overt mechanical momentum per unit length of the toroid is,

$$\begin{aligned} \mathbf{P}_{\mathrm{toroid,overt}} &= \int d\mathrm{Area} \int \frac{d\mathbf{p}_{\mathrm{overt}}}{dt}\, dt = \int \mathbf{M}_{\mathrm{initial}} \times \frac{\mathbf{E}}{c}\, d\mathrm{Area} = \int_a^b M\, \hat{\boldsymbol{\phi}} \times \frac{E_0 a\, \hat{\boldsymbol{\varrho}}}{c\varrho}\, 2\pi\varrho\, d\varrho \\ &= -\frac{2\pi E_0 M a(b-a)}{c}\, \hat{\mathbf{z}} = -\mathbf{P}_{\mathrm{inner\ conductor}} \,. \end{aligned} \tag{16}$$

Hence, the final, total momentum, $\mathbf{P}_{\mathrm{inner\ conductor}} + \mathbf{P}_{\mathrm{toroid,overt}}$, of the system is zero, as expected.

2.5 Forces and Momenta If the Electric Field Goes to Zero

If the electric field of the cylindrical capacitor drops to zero but the magnetization of the toroid remains constant, then according to eq. (15) there is no change in the "overt" mechanical momentum of the toroid, which therefore remains at rest as its "hidden" mechanical momentum drops to zero.

Meanwhile, the charges on the conductors of the cylindrical capacitor experience no axial electric field as the radial electric field drops to zero, so the conductors remain at rest.

In the final state, with zero electric field and nonzero $\mathbf{B}$ and $\mathbf{M}$ inside the toroid, there is no mechanical momentum anywhere, "hidden" or "overt", and the electromagnetic field momentum is also zero.

[6] This argument was made implicitly by Shockley [5], and explictly on p. 53 of [13].

2.6 Laboratory Demonstration of "Hidden" Mechanical Momentum

The force density (15) is radial in the present example, so its volume integral vanishes, with the implication that the mechanical momentum of the toroid remains constant as the magnetization drops to zero. If the toroid is free to move, its final velocity is in the $-z$ direction. Hence, the appearance of the final, overt mechanical momentum of the toroid can be regarded as evidence of the initial, "hidden" mechanical momentum. This suggests that a laboratory demonstration of the present example would be useful in convincing skeptics of the existence of "hidden" mechanical momentum.

So, we consider some numbers for a possible demonstration experiment. We take $a \approx b \approx 1$ cm. A practical voltage across the 1-cm cylindrical capacitor might be around 1000 V = 3.3 statvolt. This voltage is also given by $V = \int_a^b E\, d\varrho \approx 4\pi P \ln 2 \approx 3P$, so the surface charge density is $P \approx 1$ statCoulomb/cm^2. The magnetic field inside a strong permanent magnet is about $B \approx 1$ T = 10,000 G = $4\pi M$, so $M \approx 1000$ in Gaussian units. Then, the final, overt momentum would be $\approx 8\pi^2 MP/c \approx 10^{-7}$ g-cm/s, and for a toroid with mass of a few grams, its final velocity would be $\approx 10^{-7}$ cm/s, too small to be observable in a simple demonstration.

A Appendix: Abraham, Minkowski and "Hidden" Momentum

As discussed in sec. 2.4, we consider that the total "hidden" mechanical momentum in the azimuthally magnetized toroid, when inside a radial electric field, is the sum of the "hidden" mechanical momenta, $\mathbf{m} \times \mathbf{E}/c$, of the magnetic dipole moments $\mathbf{m}$ that comprise the magnetization. This nonzero mechanical momentum of the system, which is "at rest", is equal and opposite to the "field-only" momentum,

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{E-B})} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d\mathrm{Vol} = \frac{l}{4\pi c} \int_a^b \frac{E_0 a}{\varrho}\, 4\pi M\, 2\pi \varrho\, d\varrho = \frac{2\pi E_0 M a (b-a) l}{c}\, \hat{\mathbf{z}}. \tag{17}$$

In 1903 Max Abraham argued [14] that the Poynting vector [15],

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H}, \tag{18}$$

which describes the flow of energy in the electromagnetic field, when divided by c^2 has the additional significance of being the density of momentum stored in the electromagnetic field,[7]

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{A})} = \frac{\mathbf{E} \times \mathbf{H}}{4\pi c} \qquad \text{(Abraham)}. \tag{19}$$

[7]J.J. Thomson wrote the electromagnetic momentum as $\mathbf{D} \times \mathbf{H}/4\pi c$ in 1891 [16] and again in 1904 [6]. This form was also used Poincaré in 1900 [17], following Lorentz' convention [18] that the force on electric charge q be written $q(\mathbf{D} + \mathbf{v}/c \times \mathbf{H})$ and that the Poynting vector is $(c/4\pi)\, \mathbf{D} \times \mathbf{H}$. For discussion of these forms, see, for example, [2].

Of course, $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ and $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$, where $\mathbf{P}$ and $\mathbf{M}$ are the densities of electric and magnetic polarization, respectively.

In 1908 Hermann Minkowski gave an alternative derivation [19] that the electromagnetic-momentum density is,[8,9]

$$\mathbf{P}_{\rm EM}^{\rm (M)} = \frac{\mathbf{D} \times \mathbf{B}}{4\pi c} \qquad \text{(Minkowksi)}, \tag{20}$$

and the debate over the merits of these two expressions continues to this day.[10] Minkowski died before adding to the debate, while Abraham published several times on it [24, 25, 26].

In the present example, where $\mathbf{H} = 0$ everywhere, and $\mathbf{D} = \mathbf{E}$, we have that,

$$\mathbf{P}_{\rm EM}^{\rm (H)} = 0, \qquad \mathbf{P}_{\rm EM}^{\rm (M)} = \mathbf{P}_{\rm EM}^{\rm (E-B)} \tag{21}$$

As the nonzero "hidden" mechanical momentum of a system "at rest", such as the present example, must be balanced by an equal and opposite field momentum, we infer that the "field-only" momentum (17), or the Minkowski momentum (20), is the true electromagnetic field momentum, in the sense of being the one that preserves momentum conservation for a system "at rest".

To distinguish between the "field-only" momentum and the Minkowski momentum, we suppose in Appendix B that the toroid has a dielectric constant such that $\mathbf{D}$ differs from $\mathbf{E}$.

B Appendix: The Toroid is also an Dielectric

If the material of the toroid has relative dielectric constant $\epsilon > 1$, then the displacement field $\mathbf{D}$ has radial component (for $|z| < l/2$),

$$D_r(a^- < \varrho < b^+) = \epsilon E_0 \frac{a}{\varrho} \,. \tag{22}$$

The surface charge densities on the inner and outer electrodes (away from their ends) are,

$$\sigma_{a^-} = \frac{D_r(\varrho = a^-)}{4\pi} = \frac{\epsilon E_0}{4\pi} \,, \qquad \sigma_{b^+} = -\frac{D_r(\varrho = b^+)}{4\pi} = -\epsilon E_0 \frac{a}{4\pi b} \,. \tag{23}$$

In addition, the inner and outer surfaces of the dielectric toroid have bound surface charge densities, $\sigma_{\rm bound} = \mathbf{P} \cdot \hat{\mathbf{n}} = (\epsilon 0 1)\mathbf{E} \cdot \hat{\mathbf{n}}$,

$$\sigma_a = -\frac{(\epsilon - 1) E_r(\varrho = a)}{4\pi} = -\frac{(\epsilon - 1) E_0}{4\pi} \,, \qquad \sigma_b = \frac{(\epsilon - 1) E_r(\varrho = b)}{4\pi} = (\epsilon - 1) E_0 \frac{a}{4\pi b} \,. \tag{24}$$

If the magnetization drops to zero, again no electric field is induced at the outer radius, while the induced field at the inner radius is $E_{{\rm ind},z} = -4\pi \dot{M}(b - a)/c$. The inner conductor takes on momentum,

$$P_{{\rm inner\ conductor},z} = \int F_z \, dt = \frac{2\pi \epsilon E_0 M a (b - a)}{c} \,, \tag{25}$$

[8] Heaviside gave the form (3) in 1891, p. 108 of [20], and a derivation (1902) essentially that of Minkowski on pp. 146-147 of [21].

[9] See also, for example, sec. 2.1 of [22].

[10] For a lengthy bibliography on this topic, see [23].

as M drops to zero, and the toroid takes on momentum,

$$P_{\mathrm{toroid},z} = -\frac{2\pi(\epsilon - 1)E_0 M a(b-a)}{c}\,, \tag{26}$$

The total momentum imparted to the capacitor and toroid is,

$$P_{\mathrm{total},z} = \frac{2\pi E_0 M a(b-a)}{c}\,, \tag{27}$$

which is that same as for the case when $\epsilon = 1$ as discussed in sections 2.2-2.3.

The initial "field-only" momentum is again,

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{E-B})} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} = \frac{l}{4\pi c}\int_a^b \frac{E_0 a}{\varrho}\, 4\pi M\, 2\pi\varrho\, d\varrho = \frac{2\pi E_0 M a(b-a)l}{c}\,\hat{\mathbf{z}}, \tag{28}$$

as in eq. (17). The $\mathbf{H}$ field is again zero everywhere, so the Abraham momentum again vanishes,

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{A})} = \int \frac{\mathbf{E}\times\mathbf{H}}{4\pi c}\, d\mathrm{Vol} = 0 \qquad \text{(Abraham)}. \tag{29}$$

The Minkowski momentum is now,

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{M})} = \int \frac{\mathbf{D}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} = \frac{2\pi\epsilon E_0 M a(b-a)l}{c}\,\hat{\mathbf{z}}. \qquad \text{(Minkowski)}. \tag{30}$$

Of the three forms for "electromagnetic" momentum, it is the "field-only" momentum that is the same as the final mechanical momentum of the system.

This leads to the view that both the Abraham and Minkowski momenta should be regarded as "pseudomomenta", which include aspects of the momentum of the media that support the fields $\mathbf{D}$ and $\mathbf{H}$.[11]

C Appendix: The Toroid is also an Electret

C.1 Toroid with Azimuthal Magnetization and Radial Polarization

The radial electric field in the above example could have been provided by the toroid itself if it were also an electret.

First, we consider the toroid also to be an electret with uniform radial electric polarization $\mathbf{P} = -E_0\,\hat{\boldsymbol{\varrho}}/4\pi$.

[11] In the optics literature, where magnetization is generally negligible, there is a tendency to consider the Abraham momentum as the true field momentum, while the Minkowski momentum is a pseudomomentum. Of course, in examples where $\mathbf{B} = \mathbf{H}$, the Abraham momentum is the same as the "field-only" momentum, which sometimes leads to the claim that the Abraham momentum is the proper one to consider. See, for example, [27].

The surface density of bound electric charge is then,

$$\sigma_e(\varrho = a) = \mathbf{P}\cdot\hat{\mathbf{n}} = -\mathbf{P}\cdot\hat{\boldsymbol{\varrho}} = \frac{E_0}{4\pi}\,, \qquad \sigma_e(\varrho = b) = -\frac{E_0}{4\pi}. \tag{31}$$

The electric field inside the long toroid, at locations not close to its ends, is approximately radial, and follows from Gauss' law as,

$$E_\varrho(a < \varrho < b) = E_0\frac{a}{\varrho}\,, \tag{32}$$

and hence the interior **D**-field is,

$$D_\varrho(a < \varrho < b) = E_\varrho + 4\pi P_\varrho = E_0\left(1 - \frac{a}{\varrho}\right) = 4\pi P\left(1 - \frac{a}{\varrho}\right). \tag{33}$$

C.2 Toroid with Radial Magnetization and Azimuthal Polarization

For possible amusement we consider a toroid as in the above example, but with radial magnetization, $\mathbf{M} = M\,\hat{\boldsymbol{\varrho}}$, and azimuthal polarization, $\mathbf{P} = P\,\hat{\boldsymbol{\phi}}$. In this case there is no free or bound electric charge densities, either in the bulk or on the surface of the toroid. Hence, there are no sources of the electric field and $\mathbf{E} = 0$ everywhere. Inside the toroid the displacement field in nonzero, $\mathbf{D}_{\rm interior} = \mathbf{E} + 4\pi\mathbf{P} = 4\pi P\,\hat{\boldsymbol{\phi}}$.

There is no bulk density of "fictitious" magnetic charge associated with the radial magnetization, but there are "fictitious" surface magnetic charge densities given by,

$$\sigma_m(\varrho = a) = -M, \qquad \sigma_m(\varrho = b) = M. \tag{34}$$

The **H**-field inside the long toroid, at locations not close to its ends, is approximately radial, and follows from Gauss' law (here $\boldsymbol{\nabla}\cdot\mathbf{H} = 4\pi\rho_m$) as,

$$H_r(a < \varrho < b) = -4\pi M\frac{a}{\varrho}\,, \tag{35}$$

and hence the interior **B**-field is,

$$B_r(a < \varrho < b) = H_r + 4\pi M_r = 4\pi M\left(1 - \frac{a}{\varrho}\right). \tag{36}$$

This case is the dual of that described in sec. A.1, with the duality relations $\mathbf{M} \leftrightarrow \mathbf{P}$, $\mathbf{E} \leftrightarrow \mathbf{H}$ and $\mathbf{D} \leftrightarrow \mathbf{B}$.

If we accept that "hidden" mechanical momentum is due to the "external" electric field **E** on the Ampèrian currents/magnetic dipoles associated with the magnetization **M**, then there is no "hidden" mechanical momentum in the example of this section. This is consistent with the "field-only" momentum $\int \mathbf{E}\times\mathbf{B}\,d\mathrm{vol}/4\pi c$ being zero. Of course, the Abraham momentum is also zero in the case, but the Minkowski momentum is nonzero and in the $-z$ direction.

D Appendix: Earlier Versions of the Toroidal-Magnet Problem

Perhaps the earlier variant of the present problem was given in 1952, when Cullwick [28] briefly mentioned a charged particle moving along the axis of a toroidal magnet as an example where electromagnetic field momentum should be considered along with the mechanical momentum of the charged particle. This example was developed more fully in sec. 17.9 of [29], where it was pointed out that if the magnetic field of the toroid is held steady while the charge moves along the axis, the toroid experiences a force while the charge does not. See also [1, 30, 31].

The "paradox" there was considered to be resolved by noting that the sum of the mechanical momentum of the charge and the electromagnetic field momentum of the system is constant in time as the charge moves. However, even when the charge is at rest, the electromagnetic field momentum of the system is nonzero. This was considered to be acceptable as late as 1965 [31]. Only in 1967, in consideration of related examples, did Penfield and Haus, pp. 214-216 of [32] and also [33], Costa de Beauregard [34], and, most explicitly, Shockley [5] note that the total momentum of a system "at rest" should be zero, and hence these examples must also contain "hidden" mechanical momentum equal and opposite to the nonzero electromagnetic field momentum.

Cullwick's example was considered again in sec. 3 of [35] (with the assumption that the azimuthal magnetic field inside the toroid was produced by currents on its surface). However, there was no awareness of "hidden" mechanical momentum in this discussion, and like the earlier discussions through 1965 it was supposed that a system "at rest" could posses nonzero total momentum. A commentary [36] correctly noted that if the surface currents were conduction currents these would be associated with surface charges that "shield" the both the surface currents and interior of the toroid from the external electric field (but not from the induced electric field if the currents drop to zero), such that there is no initial "hidden" mechanical momentum (and no initial electromagnetic field momentum). Then, in this case there is indeed zero total momentum when the system is "at rest.[12]

The present example of an azimuthally magnetized toroid inside a cylindrical capacitor illustrates the roles of the Poynting vector, the electromagnetic field momentum and the "hidden" mechanical momentum in a system that is initially "at rest", with the pedagogic advantage that the various fields and momenta are readily computed and are relevant to the Abraham-Minkowski debate.

[12]The commentary did not point out that if the electric charge is initially at rest and the currents drop to zero, the induced electric field acts on these surface currents to give the toroid momentum equal and opposite to that of the charge.

E Appendix: Tellegen's Variant of Poynting's Theorem

The "field-only" electromagnetic momentum (17) is associated with the "field-only" Poynting vector, $\mathbf{S}^{(\mathrm{E-B})}$,

$$\mathbf{p}_{\mathrm{EM}}^{(\mathrm{E-B})} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c} = \frac{\mathbf{S}^{(\mathrm{E-B})}}{c^2}\,, \tag{37}$$

which obeys the version of Poynting's theorem that,

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{S}^{(\mathrm{E-B})} + \frac{\partial u^{(\mathrm{E-B})}}{\partial t} = P_{\mathrm{source}} &= -\frac{4\pi}{c}\mathbf{E}\cdot\mathbf{J}_{\mathrm{total}} = -\frac{4\pi}{c}\mathbf{E}\cdot\left(\mathbf{J}_{\mathrm{free}} + \frac{\partial\mathbf{P}}{\partial t} + c\boldsymbol{\nabla}\times\mathbf{M}\right), \\ \mathbf{S}^{(\mathrm{E-B})} &= \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}, \qquad u^{(\mathrm{E-B})} = \frac{E^2+B^2}{8\pi}\,.\end{aligned} \tag{38}$$

This version of Poynting's theorem was presented by Fano, Chu and Adler (1960) in sec. 7.10 of [37]. However, those authors were immediately disconcerted by the implication (their sec. 5.4) that a (nonconducting) permanent magnet in an electric field supports a flow of energy from one part of the magnet to another, even when the system is nominally static. That is, Fano, Chu and Adler were bothered by examples like the present, and this concern was perhaps the direct cause of Shockley's development of the notion of "hidden" momentum.

The discomfort of Fano, Chu and Adler with eq. (38) led to the suggestion by Tellegen of another form,

$$\boldsymbol{\nabla}\cdot\mathbf{S}^{(\mathrm{Poynting})} + \frac{\partial u^{(\mathrm{E-B})}}{\partial t} = -\frac{4\pi}{c}\mathbf{E}\cdot\left(\mathbf{J}_{\mathrm{free}} + \frac{\partial\mathbf{P}}{\partial t}\right) + 4\pi\mathbf{M}\cdot\frac{\partial\mathbf{B}}{\partial t}\,, \qquad \mathbf{S}^{(\mathrm{Poynting})} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}, \tag{39}$$

in eq. (4) of [38], which was also used in eq. (7.70) of [32], and in eq. (5) of [39]. This hybrid form of Poynting's theorem was perhaps devised so that static examples of permanent magnets in an electric field have no energy flow. However, this leaves unresolved that if the magnetization drops to zero, the sources of the electric field receive a kick, while the magnet appears not to, which is a violation of conservation of momentum.

It appears that around 1960 (shortly after the first artificial satellite, Sputnik, was launched), some people were hopeful of exotic means of rocket propulsion, and that "bootstrap spaceships" [40], based on violation of conservation of momentum, were taken seriously for a few years (and still are by a small minority such as [35]), even though these had been argued against by Slepian around 1950 in two delightful puzzlers [41].[13]

Acknowledgment

The author thanks David Griffiths and Vladimir Hnizdo for e-discussions on this note.

[13]For a review of this theme, se [42].

References

[1] K.T. McDonald, *Cullwick's Paradox: Charged Particle on the Axis of a Toroidal Magnet* (June 4, 2006), http://kirkmcd.princeton.edu/examples/cullwick.pdf

[2] K.T. McDonald, *Poynting's Theorem with Magnetic Monopoles* (Mar. 23, 2013), http://kirkmcd.princeton.edu/examples/poynting.pdf

[3] O. Costa De Beauregard, *To Believe or Not Believe in the 4-Potential, That's a Question. The Electric Helmholtz-Mikhailov Effect and its Magnetic Analog*, Found. Phys. **22**, 1485 (1992), http://kirkmcd.princeton.edu/examples/EM/costa_de_beauregard_fp_22_1485_92.pdf

[4] K.T. McDonald, *Forces on Magnetic Dipoles* (Oct. 26, 2014), http://kirkmcd.princeton.edu/examples/neutron.pdf

[5] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents"*, Phys. Rev. Lett. **18**, 876 (1967), http://kirkmcd.princeton.edu/examples/EM/shockley_prl_18_876_67.pdf

[6] J.J. Thomson, *On Momentum in the Electric Field*, Phil. Mag. **8**, 331 (1904), http://kirkmcd.princeton.edu/examples/EM/thomson_pm_8_331_04.pdf

[7] K.T. McDonald, *J.J. Thomson and "Hidden" Momentum* (Apr. 30, 2014), http://kirkmcd.princeton.edu/examples/thomson.pdf

[8] D. Babson *et al.*, *Hidden momentum, field momentum, and electromagnetic impulse*, Am. J. Phys. **77**, 826 (2009), http://kirkmcd.princeton.edu/examples/EM/babson_ajp_77_826_09.pdf

[9] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012), http://kirkmcd.princeton.edu/examples/hiddendef.pdf

[10] S. Coleman and J.H. Van Vleck, *Origin of "Hidden Momentum" Forces on Magnets*, Phys. Rev. **171**, 1370 (1968), http://kirkmcd.princeton.edu/examples/EM/coleman_pr_171_1370_68.pdf

[11] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621 (1969), http://kirkmcd.princeton.edu/examples/EM/furry_ajp_37_621_69.pdf

[12] D.J. Griffiths, *Dipoles at rest*, Am. J. Phys. **60**, 979 (1992), http://kirkmcd.princeton.edu/examples/EM/griffiths_ajp_60_979_92.pdf

[13] M.G. Calkin, *Lagrangian and Hamiltonion Mechanics* (World Scientific, 1996), kirkmcd.princeton.edu/examples/mechanics/calkin_96.pdf

[14] M. Abraham, *Prinzipien der Dynamik des Elektrons*, Ann. Phys. **10**, 105 (1903), http://physics.princeton.edu/~mcdonald/examples/EM/abraham_ap_10_105_03.pdf

[15] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),
http://kirkmcd.princeton.edu/examples/EM/poynting_ptrsl_175_343_84.pdf

[16] J.J. Thomson, *On the Illustration of the Properties of the Electric Field by Means of Tubes of Electrostatic Induction*, Phil. Mag. **31**, 149 (1891),
http://kirkmcd.princeton.edu/examples/EM/thomson_pm_31_149_91.pdf

[17] H. Poincaré, *La Théorie de Lorentz et la Principe de Réaction*, Arch. Neer. **5**, 252 (1900), http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00.pdf
Translation: *The Theory of Lorentz and the Principle of Reaction*,
http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00_english.pdf

[18] H.A. Lorentz, *La Théorie Électromagnétique Maxwell et Son Application aux Corps Mouvants*, (Brill, Leiden, 1892),
http://kirkmcd.princeton.edu/examples/EM/lorentz_theorie_electromagnetique_92.pdf

[19] H. Minkowski, *Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körper*, Nachr. Ges. Wiss. Göttingen **1**, 53 (1908),
http://kirkmcd.princeton.edu/examples/EM/minkowski_ngwg_53_08.pdf
http://kirkmcd.princeton.edu/examples/EM/minkowski_ngwg_53_08_english.pdf

[20] O. Heaviside, *Electromagnetic Theory*, Vol. 1 (Electrician Publishing, 1893),
http://kirkmcd.princeton.edu/examples/EM/heaviside_electromagnetic_theory_1.pdf

[21] O. Heaviside, *Electromagnetic Theory*, Vol. 3 (Electrician Publishing, 1912),
http://kirkmcd.princeton.edu/examples/EM/heaviside_electromagnetic_theory_3.pdf

[22] P. Milonni and R.W. Boyd, *Momentum of Light in a Dielectric Medium*, Adv. Opt. Phot. **2**, 519 (2010), http://kirkmcd.princeton.edu/examples/EM/milonni_aop_2_519_10.pdf

[23] K.T. McDonald, *Bibliography on the Abraham-Minkowski Debate* (Feb. 17, 2015),
http://kirkmcd.princeton.edu/examples/ambib.pdf

[24] M. Abraham, *Zur Elektrodynamik bewegten Körper*, Rend. Circ. Matem. Palermo **28**, 1 (1909), http://kirkmcd.princeton.edu/examples/EM/abraham_rcmp_28_1_09.pdf
http://kirkmcd.princeton.edu/examples/EM/abraham_rcmp_28_1_09_english.pdf

[25] M. Abraham, *Sull'Elettrodinamica di Minkowski*, Rend. Circ. Matem. Palermo **30**, 33 (1910), http://kirkmcd.princeton.edu/examples/EM/abraham_rcmp_30_33_10.pdf
http://kirkmcd.princeton.edu/examples/EM/abraham_rcmp_30_33_10_english.pdf

[26] M. Abraham, *Zur Frage der Symmetrie des elektromagnetischen Spannungstensors*, Z. Phys. **17**, 537 (1914),
http://kirkmcd.princeton.edu/examples/EM/abraham_zp_15_537_14_english.pdf

[27] D.J. Griffiths, *Comment on "Electromagnetic momentum conservation in media"*, Ann. Phys. 327, 1290 (2012),
http://kirkmcd.princeton.edu/examples/EM/griffiths_ap_327_1290_12.pdf

[28] E.G. Cullwick, *Electromagnetic Momentum and Newton's Third Law*, Nature **170**, 425 (1952), http://kirkmcd.princeton.edu/examples/EM/cullwick_nature_170_425_52.pdf

[29] E.G. Cullwick, *Electromagnetism and Relativity* (Longmans, Green, 1959), pp. 232-238, http://kirkmcd.princeton.edu/examples/EM/cullwick_ER_59.pdf

[30] G.T. Trammel, *Aharonov-Bohm Paradox*, Phys. Rev. **134**, B1183 (1964), http://kirkmcd.princeton.edu/examples/EM/trammel_pr_134_B1183_64.pdf

[31] T.T. Taylor, *Electrodynamic Paradox and the Center-of-Mass Principle*, Phys. Rev. **137**, B467 (1965), http://kirkmcd.princeton.edu/examples/EM/taylor_pr_137_B467_65.pdf

[32] P. Penfield, Jr. and H.A. Haus, *Electrodynamics of Moving Media*, p. 215 (MIT, 1967), http://kirkmcd.princeton.edu/examples/EM/penfield_haus_chap7.pdf

[33] H.A. Haus and P. Penfield Jr, *Forces on a Current Loop*, Phys. Lett. **26A**, 412 (1968), http://kirkmcd.princeton.edu/examples/EM/penfield_pl_26a_412_68.pdf

[34] O. Costa de Beauregard, *A New Law in Electrodynamics*, Phys. Lett. **24A**, 177 (1967), http://kirkmcd.princeton.edu/examples/EM/costa_de_beauregard_pl_24a_177_67.pdf
Physical Reality of Electromagnetic Potentials?, Phys. Lett. **25A**, 375 (1967), http://kirkmcd.princeton.edu/examples/EM/costa_de_beauregard_pl_25a_375_67.pdf
"Hidden Momentum" in Magnets and Interaction Energy, Phys. Lett. **28A**, 365 (1968), http://kirkmcd.princeton.edu/examples/EM/costa_de_beauregard_pl_28a_365_68.pdf

[35] J.M. Aguirregabiria, A Hernández and M Rivas, *Linear momentum density in quasistatic electromagnetic systems*, Eur. J. Phys. **25**, 255 (2004), http://kirkmcd.princeton.edu/examples/EM/aguirregabiria_ejp_25_555_04.pdf

[36] A.H. Kholmeiskii, *Comment on the paper 'Linear momentum density in quasistatic electromagnetic systems'*, Eur. J. Phys. **26**, L21 (2005), http://kirkmcd.princeton.edu/examples/EM/kholmetskii_ejp_26_L21_05.pdf

[37] R.M. Fano, L.J. Chu and R.B. Adler, *Electromagnetic Fields, Energy and Forces* (Wiley, 1960), kirkmcd.princeton.edu/examples/EM/fano_chu_adler_60.pdf

[38] B.D.H. Tellegen, *Magnetic-Dipole Models*, Am. J. Phys. **30**, 650 (1962), http://kirkmcd.princeton.edu/examples/EM/tellegen_ajp_30_650_62.pdf

[39] E. Engels and F. Pauwels, *Interpretation of Magnetic Energy Density*, Appl. Sci. Res. **29**, 14 (1974), http://kirkmcd.princeton.edu/examples/EM/engels_asr_29_14_74.pdf

[40] O. Costa de Beauregard, *Magnetodynamic Effect*, Nuovo Cim. **B63**, 611 (1969), http://kirkmcd.princeton.edu/examples/EM/costa_de_beauregard_nc_b63_611_69.pdf

[41] J. Slepian, *Electromagnetic Space-Ship*, Electrical Engineering **68**, 145, 245 (1949), http://kirkmcd.princeton.edu/examples/EM/slepian_ee_68_145_49
Electrostatic Space Ship, Electrical Engineering **69**, 164, 247 (1950), http://kirkmcd.princeton.edu/examples/EM/slepian_ee_69_164_50.pdf

[42] K.T. McDonald, *No Bootstrap Spaceships* (Aug. 9, 2018),
http://kirkmcd.princeton.edu/examples/bootstrap.pdf