Backtracking of Charged Particles in a Magnetic Field

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1 Problem

Suppose that a particle of electric charge q follows a trajectory $\mathbf{r}(t)$ in a known, static magnetic field $\mathbf{B}(\mathbf{r})$. For what conditions could a different particle follow the same trajectory, but with velocity opposite to that of the original particle.

Consider also special cases in which the magnetic field obeys a spatial symmetry.

2 Solution

The equation of motion for the charged particle is (in SI units, and ignoring forces other than that due to the magnetic field),

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = q\mathbf{v} \times \mathbf{B}, \tag{1}$$

where $\mathbf{v} = d\mathbf{r}/dt$ is the particle's velocity, $\mathbf{p} = \gamma m\mathbf{v} = m\mathbf{v}/\sqrt{1 - v^2/c^2}$ is its momentum, m is its rest mass and c is the speed of light in vacuum. If the velocity $\mathbf{v}$ were reversed at some time, the sign of the Lorentz force (1) would be reversed, and the particle would not retrace its earlier trajectory. However, the electromagnetic interaction is time-reversal invariant, so if the velocities of all particles in the Universe were reversed, the magnetic field would change sign, as well as the velocity of the test particle, and the trajectory of the latter would be unchanged.

In the rest of this note we consider scenarios in which the magnetic field is not changed.

In this case, an antiparticle of mass m, charge $-q$, but with its velocity reversed with respect to the original particle would "backtrack" along the original trajectory.[1,2]

2.1 Retracking When the Magnetic Field Has a Mirror Symmetry

We now discuss a peculiar result that holds when the magnetic field has a mirror symmetry about, say, the y-z plane,

$$B_x(-x, y, z) = -B_x(x, y, z), \quad B_y(-x, y, z) = B_y(x, y, z), \quad B_z(-x, y, z) = B_z(x, y, z). \tag{2}$$

In particular, this symmetry holds for an axially symmetric field,

$$\mathbf{B} = B_\rho(\rho, z)\,\hat{\boldsymbol{\rho}} + B_z(\rho, z)\,\hat{\mathbf{z}} = B_\rho(\rho, z)\frac{x}{\rho}\,\hat{\mathbf{x}} + B_\rho(\rho, z)\frac{y}{\rho}\,\hat{\mathbf{y}} + B_z(\rho, z)\,\hat{\mathbf{z}}, \tag{3}$$

[1] As remarked by Feynman [1], an antiparticle moving forward in time is equivalent to a particle moving backwards in time.

[2] Touschek (1960) was perhaps the first to realize that counterpropagating beams of electrons and positrons could circulate in a single (storage) ring of magnets [2].

where $\rho = \sqrt{x^2 + y^2}$. Then, the Lorentz force (1) obeys,

$$\frac{F_x}{q} = v_y B_z(x, y, z) - v_z B_y(x, y, z) = -[(-v_y) B_z(-x, y, z) - (-v_z) B_y(-x, y, z)], \tag{4}$$

$$\frac{F_y}{q} = v_z B_x(x, y, z) - v_x B_z(x, y, z) = (-v_z) B_x(-x, y, z) - v_x B_z(-x, y, z), \tag{5}$$

$$\frac{F_z}{q} = v_x B_y(x, y, z) - v_y B_x(x, y, z) = v_x B_y(-x, y, z) - (-v_y) B_x(-x, y, z)]. \tag{6}$$

This has the implication that if the charge q is subject to the transformation at some time t,

$$x \to -x, \qquad y \to y, \qquad z \to z, \qquad v_x \to v_x, \qquad v_y \to -v_y, \qquad v_z \to -v_z, \tag{7}$$

then the Lorentz force transforms according to,

$$F_x \to -F_x, \qquad F_y \to F_y, \qquad F_z \to F_z. \tag{8}$$

The trajectory of the transformed charge is then the mirror image in the y-z plane of the original trajectory.

And if at some later time, the transformation (7) is applied to the previously transformed particle, it will then again lie on the original trajectory, and as time advances it will retrace a portion of the original trajectory. We might call this procedure "retracking" rather than "backtracking".

This double transformation is not practical in the laboratory, but it is suitable for use in a computer simulation of an earlier portion of a particle's trajectory (without reversing the charge or the magnetic field or the direction of time). Thanks to Xiaoping Ding for pointing out this intricate result.

Note that "retracking" could also be accomplished by two transformations, at different times, in which the charge q and the velocity $\mathbf{v}$ of the particle are reversed, but its position is not changed.

References

[1] R.P. Feynman, *Theory of Positrons*, Phys. Rev. **76**, 749 (1949),
http://kirkmcd.princeton.edu/examples/QED/feynman_pr_76_749_49.pdf

[2] L. Bonolis and G. Pancheri, *Bruno Touschek: particle physicist and father of the e^+e^- collider*, Eur. Phys. J. H **36**, 1 (2011),
http://kirkmcd.princeton.edu/examples/accel/bonolis_epjh_36_1_11.pdf