Energy Flow in a Transformer

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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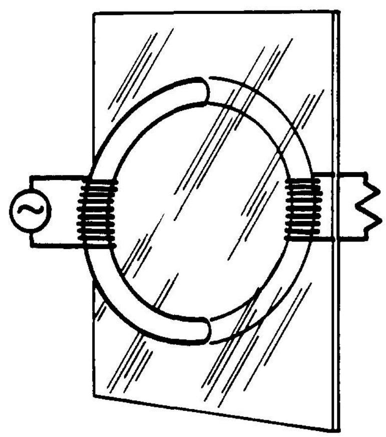
1 Problem

Discuss the flow of energy from voltage source to load resistor in a circuit that contains a transformer. As an application, consider wireless power transmission.

2 Solution

This problem was posed by Siegman [1], which led to several conflicting responses [2, 3, 4, 5], all of which seem somewhat misguided.[1]

The challenge is to explain the flow of energy from an AC source to a load resistor in a circuit in which the source and load are isolated by a transformer, as sketched below (from [1]).

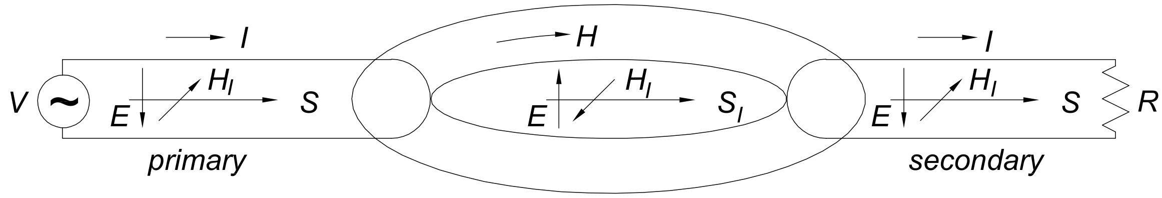


We argue that the energy flow bypasses the transformer, whereas the respondents [2, 3, 4, 5] tried to implement variants of the "plumbing analogy" in which the transformer windings and/or core participated in the transport of energy.

2.1 Transformer with a Highly Permeable, Nonconducting Core

Following the spirit of Siegman's sketch (shown above), we consider a transformer in which the toroidal core is of a highly permeable material (relative permeability $\mu \gg 1$). We also suppose that the core is nonconducting (although most highly permeable materials are conductors). Further, we suppose that the primary and secondary windings both consist of only a single turn (which is sufficient for good transformer action if μ is very large). A sketch of such a transformer, with AC voltage source V and load resistor R is shown on the next page.

[1] An earlier version [6] of this problem also led to a somewhat unsatisfactory discussion [7].



For a circuit analysis we need to know the self inductances L_p and L_s of the primary and secondary, as well as the mutual inductance M. One advantage of a highly permeable core is that $L_p \approx L_s \approx M$ even when the primary and secondary windings are on opposite sides of the core. Then the primary and secondary currents are given by,

$$I_p = V\left(\frac{1}{R} - \frac{i}{\omega L_p}\right), \qquad I_s = \frac{V}{R}, \tag{1}$$

where ω is the angular frequency of the voltage source. Since L_p is large when a highly permeable core is used, the condition $\omega L_p \gg R$ is readily satisfied, in which case $I_P \approx I_s = V/R$, and the time-average power delivered by the source is $\approx V^2/2R$. The energy stored in the magnetic (and electric) fields of the circuit oscillates, but the power associated with these changing fields is small compared to the power dissipated in the load resistor.

The Poynting vector in the region between the voltage source and the core, and between the core and the load resistor are readily understood by thinking of these portions of the circuit as two-wire (Lecher) transmission lines. When the current in the upper conductors is to the right, as shown above, the magnetic field between the wires is into the paper, and the electric field between the wires is downwards. Hence, the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is to the right in the vicinity of the transmission lines.

The challenge of Siegman is to characterize the Poynting vector in the core region. Inside the toroidal core the changing magnetic fields $\mathbf{B} = \mu\mu_0\mathbf{H}$ is associated with a (changing) electric field that circulates around its circular centerline, such that $\mathbf{S}$ points inwards towards this circle when B is increasing. This Poynting flux is associated with changing the magnetic field energy of the torus, rather than flow of energy from the source to the load resistor.

Outside the torus, an electric field is induced by the changing magnetic field inside the torus, and again this field circulates about the circular centerline of the torus, such that in the region between the primary and secondary the electric field is upwards. In this region, where the relative permeability is unity, $\mathbf{B} = \mu_0\mathbf{H}$ and $\mathbf{H}$ can be deduced from the conduction currents in the primary and secondary as well as the "displacement current" $\partial\mathbf{D}/\partial t$. The magnetic field due to the conduction currents is shown as $\mathbf{H}_I$ in the above figure, and points into the page, while the magnetic field $\mathbf{H}_D$ associated with the "displacement current" is azimuthal about the symmetry axis of the core (and is not shown in the figure). The Poynting vector $\mathbf{S}_I$ associated with the fields $\mathbf{E}$ and $\mathbf{H}_I$ points to the right and carries energy from the primary into the secondary; the Poynting vector associated with the fields $\mathbf{E}$ and $\mathbf{H}_D$ is radial and relates to the flow of energy stored in the "near zone" of the core.

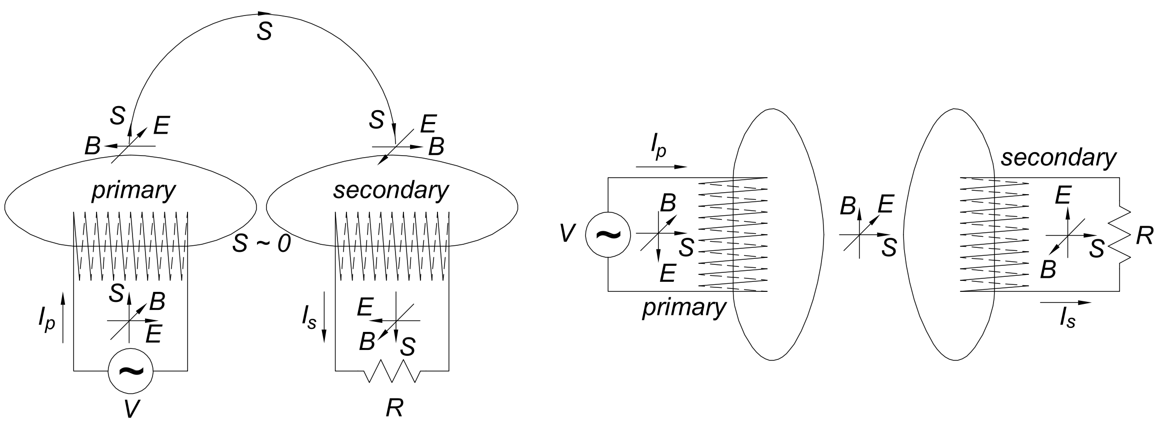
2.2 Air-Core Transformer with Full-Azimuth Windings

An extreme variant of a toroidal transformer is that it have an air core (no permeable material) and multiturn windings that cover its entire surface The helical windings[2] have a current component that goes around the major circumference of the torus, such that the fields outside the toroid have essentially the same character as those for the example of a highly permeable core. The Poynting vector associated with the magnetic field of this current again flows across a diameter of the toroid from the vicinity of the primary leads to that of the secondary leads.

2.3 Two Separated Circuits with Mutual Inductance

Another variant of the present problem is the case of two circuits that are physically distinct but which have a mutual inductance due to, say, coils in each circuit.

A common view is that the circuits are coupled because the magnetic flux from the "primary" coil is linked by the "secondary" coil. This suggests that the coils in the two circuits should be arranged coaxially, as in the left figure below.[3] In this configuration the flow of energy between the two circuits that is "roundabout", since the energy flow vector (Poynting vector) $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ does not follow lines of the magnetic field $\mathbf{H}$ (or lines of the electric field $\mathbf{E}$). In particular, the Poynting vector is zero along the common axis of the two coils.



More direct flow of energy from one circuit to the other is obtained when the coils have their axes parallel, and both axes perpendicular to the line of centers of the two coils, as shown in the above right figure. In this case the Poynting vector can flow more or less directly from one circuit to the other. This configuration is well known in the antenna community, where the two circuits are many wavelengths apart, and mutual inductance is no longer a relevant concept.

[2]If the windings go around the entire toroid helically and then wind back to the starting point there will be no net azimuthal current (around the symmetry axis of the toroid, and we speak of a "counterwound helical toroid" [8]. This winding geometry has little advantage for a transformer, but is used in certain antennas.

[3]This configuration is used in wireless power transmission. See, for example, [9].

References

[1] A.E. Siegman, *Letter to the Editor*, Am. J. Phys. **51**, 492 (1983),
http://kirkmcd.princeton.edu/examples/EM/siegman_ajp_51_492_83.pdf

[2] W.A. Newcomb, *Where is the Poynting vector in an ideal transformer?*, Am. J. Phys. **52**, 723 (1984), http://kirkmcd.princeton.edu/examples/EM/newcomb_ajp_52_723_84.pdf
Newcomb proposes a magnetic field $\mathbf{B}$ in his Fig. 1 that does not obey $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$.

[3] P. Lorrain, *The Poynting vector in an ideal transformer*, Am. J. Phys. **52**, 987 (1984),
http://kirkmcd.princeton.edu/examples/EM/lorrain_ajp_52_987_84.pdf

[4] F. Herrmann and G.B. Schmid, *The Poynting vector field and energy flow within a transformer*, Am. J. Phys. **54**, 528 (1986),
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[5] N. Morton, *The Poynting vector distribution in a simple transformer*, Am. J. Phys. **55**, 472 (1987), http://kirkmcd.princeton.edu/examples/EM/morton_ajp_55_472_87.pdf

[6] W. Richter, *Electrical Essay for Recreation*, Elec. Eng. **66**, 44 (1947),
http://kirkmcd.princeton.edu/examples/EM/richter_ee_66_44_47.pdf

[7] J. Slepian, *Energy Flow*, Elec. Eng. **66**, 318 (1947),
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[8] K.T. McDonald, *Electromagnetic Fields of a Small Helical Toroidal Antenna* (Dec. 27, 2013), http://kirkmcd.princeton.edu/examples/cwhta.pdf

[9] K.T. McDonald, *Wireless Power Transmission* (Dec. 27, 2013),
http://kirkmcd.princeton.edu/examples/wireless.pdf