Transition Radiation at a Metal-Vacuum Interface and at an Interface between Two Dielectrics

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(December 8, 2021)

1 Problem

A particle of charge e with velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$ passes through a metallic beam window at $z = 0$ and emerges into vacuum for $z > 0$. What is the frequency-angle spectrum of the radiation in the region $z > 0$, assuming that the beam window is perfectly conducting and an infinite sheet?

Consider also a charge e that moves through two semi-infinite media, with dielectric constants ϵ_1 for $z < 0$ and $\epsilon_2 > \epsilon_1$ for $z > 0$, on trajectory $\mathbf{r} = vt\,\hat{\mathbf{z}}$ where $v < c/n_2 = c/\sqrt{\epsilon_2} < c/n_1$, where c is the speed of light in vacuum, and $n = \sqrt{\epsilon}$ is the index of refraction of a dielectric (that has unit relative permeability).

2 Solution

We consider a method [1, 2] that worked well in characterizing Čerenkov radiation, based on an expression for the spectrum of energy *vs.* angular frequency ω and solid angle Ω of a pulse of radiation due to electric charge e with (generally time-dependent) velocity $\mathbf{v}$,[1]

$$\frac{dU_\omega}{d\Omega} = \frac{e^2\omega^2 n}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \boldsymbol{\beta})\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\, dt\right|^2 = \frac{\omega^2 n}{4\pi^2 c}\left|\int_{-\infty}^{\infty} e\boldsymbol{\beta} \times \hat{\mathbf{k}}\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\, dt\right|^2, \qquad (1)$$

in Gaussian units, and where $\mathbf{k}$ is the wave vector with $k = n\,\omega/c$ in case of a medium with index of refraction n, and hence $\hat{\mathbf{k}} = \hat{\mathbf{n}}$ is the unit vector pointing to the observer. Also, $\boldsymbol{\beta} = \mathbf{v}/c$.

2.1 Metal-Vacuum Interface

2.1.1 A First Approximation

We first apply eq. (1) simply to the motion of charge e with position $\mathbf{r} = vt\,\hat{\mathbf{z}} = \beta ct\,\hat{\mathbf{z}}$ for constant velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$ for $z > 0$ in vacuum, where $\mathbf{k} = \omega\,\hat{\mathbf{k}}/c = \omega\,(\sin\theta, 0, \cos\theta)/c$ for an observer in the x-z plane,

$$\frac{dU_\omega}{d\Omega} = \frac{\omega^2}{4\pi^2 c}\left|\int_0^{\infty} e\beta\,\hat{\mathbf{z}} \times \hat{\mathbf{k}}\, e^{i(\omega t - \mathbf{k}\cdot vt\,|\hat{\mathbf{z}})}\, dt\right|^2 = \frac{e^2\beta^2\omega^2}{4\pi^2 c}\left|\int_0^{\infty} \sin\theta\, e^{i\omega t(1-\beta\cos\theta)}\, dt\right|^2$$

$$= \frac{e^2\beta^2\omega^2\sin^2\theta}{4\pi^2 c}\left|\frac{e^{i\omega\infty(1-\beta\cos\theta)} - 1}{i\omega(1-\beta\cos\theta)}\right|^2 = \frac{e^2\beta^2\sin^2\theta}{4\pi^2 c(1-\beta\cos\theta)^2} \qquad (2)$$

[1]This approach follows pp. 261-265 of [3]. See also sec. 20-7 of [4].
A derivation of eq. (1) in vacuum via the Liènard-Wiechert fields is given in sec. 14.5 of [5].

$$U_\omega = \int \frac{dU_\omega}{d\Omega}\, d\Omega = \frac{e^2\beta^2}{2\pi c}\int_0^1 \frac{1-\cos^2\theta}{(1-\beta\cos\theta)^2}\, d\cos\theta$$

$$= \frac{e^2\beta^2}{2\pi c}\left[\frac{1}{\beta(1-\beta\cos\theta)} + \frac{1}{\beta^3}\left(1-\beta\cos\theta - 2\ln(1-\beta\cos\theta) - \frac{1}{1-\beta\cos\theta}\right)\right]_0^1$$

$$= \frac{e^2\beta^2}{2\pi c}\left[\frac{1}{1-\beta^2} + \frac{1}{\beta^3}\left(-\beta - \ln\frac{1-\beta}{1+\beta} - \frac{\beta}{1-\beta^2}\right)\right]$$

$$= \frac{e^2\beta^2}{2\pi c}\left[\frac{1}{1-\beta^2}\left(1-\frac{1}{\beta^2}\right) - \frac{1}{\beta^2} + \frac{1}{\beta^3}\ln\frac{1+\beta}{1-\beta}\right] = \frac{e^2}{2\pi c}\left[\frac{2}{\beta}\ln\gamma(1+\beta) - 1\right], \qquad (3)$$

taking $e^{i\infty} = 0$ as representing the time-average of the oscillatory function, and using Dwight 90.2 and 92.2 [6]. In the relativistic limit, $\beta \to 1$, where most observations of transition radiation have been made,

$$U_\omega \to \frac{e^2\ln 2\gamma}{\pi c}, \qquad \text{with} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}. \qquad (4)$$

As $\beta \to 0$, eq. (3) goes to the nonzero value $e^2/2\pi c$.

2.1.2 A Better Approximation – Image Method

The preceding analysis ignored the effect of the time-dependent charge density induced on the surface of the conducting sheet at $z = 0$. In a better approximation we suppose that this effect is equivalent to the presence of an image charge $-e$ at $z = -vt$ for $t = 0$ (and the conducting sheet is absent. For the actual situation, the surface charge at time $t > 0$ exists on the sheet only inside a circle of radius ct about the origin, which the image method implies that the surface charge density is nonzero everywhere on the sheet for $t > 0$, so the result of this section is not "exact".

Adapting eq. (1) to include the image charge, we have,[2]

$$\frac{dU_\omega}{d\Omega} = \frac{\omega^2}{4\pi^2 c}\left|\int_0^\infty e\,\boldsymbol{\beta}_e \times \hat{\mathbf{k}}\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r}_e)}\, dt + \int_0^\infty (-e)\,\boldsymbol{\beta}_{-e}\times\hat{\mathbf{k}}\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r}_{-e})}\, dt\right|^2$$

$$= \frac{e^2\omega^2\beta^2\sin^2\theta}{4\pi^2 c}\left|\int_0^\infty e^{i\omega t(1-\beta\cos\theta)}\, dt + \int_0^\infty e^{i\omega t(1+\beta\cos\theta)}\, dt\right|^2 \qquad (5)$$

$$= \frac{e^2\omega^2\beta^2\sin^2\theta}{4\pi^2 c}\left|\frac{e^{i\omega\infty(1-\beta\cos\theta)}-1}{i\omega(1-\beta\cos\theta)} + \frac{e^{i\omega\infty(1+\beta\cos\theta)}-1}{i\omega(1+\beta\cos\theta)}\right|^2 = \frac{e^2\beta^2}{\pi^2 c}\frac{\sin^2\theta}{1-\beta^2\cos^2\theta},$$

where $\mathbf{k} = \omega\,\hat{\mathbf{k}}/c = \omega(\sin\theta, 0, \cos\theta)/c$ is in the direction of the radiation to the observer (located at large $z > 0$), $\hat{\mathbf{k}} = \hat{\mathbf{n}}$, $\boldsymbol{\beta}_{e,-e} = \pm v\,\hat{\mathbf{z}}/c = \pm\beta\,\hat{\mathbf{z}}$, $\mathbf{r}_{e,-e} = \pm vt\,\hat{\mathbf{z}} = \pm\beta ct\,\hat{\mathbf{z}}$, and we take $e^{i\omega\infty(1\pm\beta\cos\theta)} = 0$, as representing the time-average of the oscillatory behavior at large times.

[2]The result (5) was first obtained in [7]. See also sec. 2.1.2 of [8].

This result was obtained by a different approximation, called " quasi-classical" in [9]. See also sec. 28b, particularly p. 283, of [10].

The frequency spectrum of the transition radiation is, noting that $0 < \theta < \pi/2$ for an observer with $z > 0$,

$$U_\omega = \int \frac{dU_\omega}{d\Omega}\, d\Omega = \frac{2e^2}{\pi c \beta^2} \int_0^1 \frac{1 - \cos^2\theta}{(1/\beta^2 - \cos^2\theta)^2}\, d\cos\theta$$

$$= \frac{2e^2}{\pi c\beta^2}\left[\frac{\beta^2\cos\theta}{2(1/\beta^2 - \cos^2\theta)} + \frac{\beta^3}{4}\ln\frac{1/\beta + \cos\theta}{1/\beta - \cos\theta} - \frac{\cos\theta}{2(1/\beta^2 - \cos^2\theta)} + \frac{\beta}{4}\ln\frac{1/\beta + \cos\theta}{1/\beta - \cos\theta}\right]_0^1$$

$$= \frac{2e^2}{\pi c\beta^2}\left[\frac{\beta^4}{2(1-\beta^4)} + \frac{\beta^3}{4}\ln\frac{1+\beta}{1-\beta} - \frac{\beta^2}{2(1-\beta^4)} + \frac{\beta}{4}\ln\frac{1+\beta}{1-\beta}\right]$$

$$= \frac{2e^2}{\pi c\beta^2}\left[-\frac{\beta^2}{2} + \frac{\beta(1+\beta^2)}{4}\ln(\gamma^2(1+\beta)^2)\right] = \frac{e^2}{\pi c}\left[\frac{1+\beta^2}{\beta}\ln\gamma(1+\beta) - 1\right], \tag{6}$$

using Dwight 140.2 and 142.2 [6]. In the relativistic limit, $\beta \to 1$,

$$U_\omega \to \frac{2e^2 \ln 2\gamma}{\pi c}, \tag{7}$$

which is twice that found in the first approximation (4). As $\beta \to 0$, eq. (6) goes to zero.

2.2 Interface between Two Dielectrics

2.2.1 A First Approximation

For a first approximation we apply a version of eq. (1) considering only the motion $\mathbf{r} = vt\,\hat{\mathbf{z}} = \beta ct\,\hat{\mathbf{z}}$ of the charge e, assumed to constant velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$ both in the medium with (relative) dielectric constant ϵ_1 at $z < 0$ and in the medium with (relative) dielectric constant ϵ_2 at $z > 0$. We consider only the forward radiation ($|\theta| < \pi/2$ received by an observer in the x-z plane with large $z > 0$. If a ray observed at angle θ_2 for $z > 0$ originated at $z < 0$ it had angle θ_1 for $z < 0$ related by Snell's law, $n_1 \sin\theta_1 = n_2 \sin\theta_2$, where we take the indices of refraction to be $n_{1,2} = \sqrt{\epsilon_{1,2}}$. To avoid complications of total internal reflection at the interface of rays emanating from $z < 0$, we suppose that $n_1 < n_2$, *i.e.*, $\epsilon_1 < \epsilon_2$. Then, noting that $\mathbf{k}_i = n_i\,\omega\,\hat{\mathbf{k}}_i/c$, and setting $e^{\pm i\infty} = 0$ as before,

$$\frac{dU_\omega}{d\Omega} = \frac{n_2\,\omega^2}{4\pi^2 c}\left|\int_{-\infty}^0 e\,\hat{\mathbf{k}}_1 \times \boldsymbol{\beta}\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\, dt + \int_0^\infty e\,\hat{\mathbf{k}}_2 \times \boldsymbol{\beta}\, e^{i(\omega t - \mathbf{k}_2\cdot\mathbf{r})}\, dt\right|^2$$

$$= \frac{e^2 n_2\,\omega^2\beta^2}{4\pi^2 c}\left|\sin\theta_1 \int_{-\infty}^0 e^{i\omega t(1 - n_1\beta\cos\theta_1)}\, dt + \sin\theta_2 \int_0^\infty e^{i\omega t(1 - n_2\beta\cos\theta_2)}\, dt\right|^2$$

$$= \frac{e^2 n_2\,\omega^2\beta^2}{4\pi^2 c}\left|\frac{n_2}{n_1}\sin\theta_2 \int_{-\infty}^0 e^{i\omega t(1 - n_1\beta\cos\theta_1)}\, dt + \sin\theta_2 \int_0^\infty e^{i\omega t(1 - n_2\beta\cos\theta_2)}\, dt\right|^2$$

$$= \frac{e^2 n_2\,\omega^2\beta^2\sin^2\theta_2}{4\pi^2 c\, n_1^2}\left|n_2 \frac{1 - e^{-i\omega\infty(1 - n_1\beta\cos\theta_1)}}{i\omega(1 - n_1\beta\cos\theta_1)} + n_1 \frac{e^{i\omega\infty(1 + n_2\beta\cos\theta_2)} - 1}{i\omega(1 - n_2\beta\cos\theta_2)}\right|^2$$

$$= \frac{e^2 n_2\,\beta^2\sin^2\theta_2}{4\pi^2 c\, n_1^2}\left|\frac{n_2}{1 - n_1\beta\cos\theta_1} - \frac{n_1}{1 - n_2\beta\cos\theta_2}\right|^2. \tag{8}$$

The transition radiation of eq. (8) is large only when the denominators in the last line are small, *i.e.*, when n_i, β and $\cos\theta_i$ are all close to 1. We recall the atomic model of the frequency dependence of the dielectric constant ϵ,[3]

$$\epsilon(\omega) = 1 + \frac{4piNe^2}{m} \sum_j \frac{f_j}{\omega_j - \omega^2 - i\Gamma_j\,\omega}\,, \qquad \sum_j f_j = 1, \tag{9}$$

where N is the number density of atoms, e and m are the charge and mass of an electron, f_j is the relative strength (oscillator strength) of oscillation j in the atom, with angular frequency ω_j and damping constant Γ_j.e The dielectric constant is near 1 only for high frequencies, in which case,

$$\epsilon(\omega \text{ large}) \approx 1 - \frac{4piNe^2}{m\,\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\,, \tag{10}$$

where $\omega_p = \sqrt{4\pi Ne^2/m}$ is the plasma frequency of the medium. The corresponding index of refraction is,[4]

$$n(\omega \text{ large}) = \sqrt{\epsilon(\omega \text{ large})} \approx 1 - \frac{\omega_p^2}{2\,\omega^2}\,. \tag{11}$$

For β near 1, we write $\beta = \sqrt{1 - 1/\gamma^2} \approx 1 - 1/2\gamma^2$, and of $\cos\theta_i$ near 1 we have $\cos\theta_i \approx 1 - \theta_i/2$. In these limits,

$$\frac{1}{1 - n_i\beta\cos\theta_i} \approx \frac{1}{1 - (1 - \omega_{p_i}^2/2\omega^2)(1 - 1/2\gamma^2)(1 - \theta_i^2/2)} \approx \frac{2}{\omega_{p_i}^2/\omega^2 + 1/\gamma^2 + \theta_i^2}\,, \tag{12}$$

and the frequency-angle spectrum of the transition radiation is, noting that $\theta_1 \approx \theta_2$ since $n_1 \approx n_2 \approx 1$,

$$\frac{dU_\omega}{d\Omega} \approx \frac{e^2\theta_2^2}{\pi^2 c} \left| \frac{1}{\omega_{p_1}^2/\omega^2 + 1/\gamma^2 + \theta_2^2} - \frac{1}{\omega_{p_2}^2/\omega^2 + 1/\gamma^2 + \theta_2^2} \right|^2 . \tag{13}$$

As expected, this vanishes if the two media are the same, *i.e.*, if $\omega_{p_1} = \omega_{p_2}$.

The angular distribution peaks for,

$$\frac{\omega_{p_1}^2}{\omega^2} + \frac{1}{\gamma^2} < \theta_2^2 < \frac{\omega_{p_2}^2}{\omega^2}\,, \tag{14}$$

i.e., in the forward direction (like Bremsstrahlung, such that there was early skepticism that transition radiation is not distinct from Bremsstrahlung).

[3] See, for example, p. 135 of [11] or sec. 7.5 of [5]

[4] At the high frequencies where the index of refraction n is less than 1, there is no Čerenkov radiation, so in practice the issue of interference between Čerenkov and transition radiation is moot.

The angular distribution can be integrated to give, with $x = \theta_2^2$ and $A_i = \omega_{p_i}^2/\omega^2 + 1/\gamma^2$,

$$U_\omega(\omega \text{ large}) = \int \frac{dU_\omega}{d\Omega}\, d\Omega \approx 2\pi \int_0^\infty \frac{dU_\omega}{d\Omega}\, \theta_2\, d\theta_2 = \pi \int_0^\infty \frac{dU_\omega}{d\Omega}\, d\theta_2^2$$
$$\approx \frac{e^2}{\pi c} \int_0^\infty x \left| \frac{1}{A_1 + x} - \frac{1}{A_2 + x} \right|^2 dx = \frac{e^2 (A_1 - A_2)^2}{\pi c} \int_0^\infty dx\, \frac{x}{(A_1 + x)^2 (A_2 + x)^2}$$
$$= \frac{e^2 (A_1 - A_2)^2}{\pi c} \left[\frac{A_1 + A_2}{(A_1 - A_2)^3} \ln \frac{A_1 + x}{A_2 + x} + \frac{1}{(A_1 - A_2)^2} \left(\frac{A_1}{A_1 + x} + \frac{A_2}{A_2 + x} \right) \right]_0^\infty$$
$$= \frac{e^2}{\pi c} \left[\frac{A_1 + A_2}{(A_1 - A_2)} \ln \frac{A_2}{A_1} - 2 \right] = \frac{e^2}{\pi c} \left[\frac{\omega_{p_1}^2 + \omega_{p_2}^2 + 2\omega^2/\gamma^2}{\omega_{p_1}^2 - \omega_{p_2}^2} \ln \left(\frac{\omega_{p_2}^2 + \omega^2/\gamma^2}{\omega_{p_1}^2 + \omega^2/\gamma^2} \right) - 2 \right]$$
$$\approx \frac{e^2}{6\pi c} \frac{\gamma^4}{\omega^4} (\omega_{p_1}^2 - \omega_{p_2}^2)^2, \qquad (15)$$

using Dwight 113,1 [6], and where the last approximation requires evaluation to third order.

The photon number spectrum is obtained by dividing eq. (15) by $\hbar$,

$$N_\omega = \frac{U_\omega}{\hbar} \approx \frac{1}{6\pi} \frac{e^2}{\hbar c} \frac{\gamma^4}{\omega^4} (\omega_{p_1}^2 - \omega_{p_2}^2)^2 = \frac{1}{6\pi} \frac{1}{137} \frac{\gamma^4}{\omega^4} (\omega_{p_1}^2 - \omega_{p_2}^2)^2, \qquad (16)$$

which is a very weak effect, although it does vary as γ^4, so can be significant for ultrarelativistic particles.

Our eq. (8) appears to be rather different than the Ginzburg-Frank result,[5]

$$\frac{dU_\omega}{d\Omega} = \frac{e^2 v^2 n_2 \sin^2\theta_2 \cos^2\theta_2}{\pi^2 c^3} \left| \frac{\epsilon_1 - \epsilon_2}{\left(1 - \epsilon_2 \beta^2 \cos^2\theta_2\right) \left(1 - \beta\sqrt{\epsilon_1 - \epsilon_2 \sin^2\theta_2}\right)} \right.$$
$$\left. \times \frac{1 - \beta^2 \epsilon_2 - \beta\sqrt{\epsilon_1 - \epsilon_2 \sin^2\theta_2}}{\epsilon_1 \cos\theta_2 + \sqrt{\epsilon_1 \epsilon_2 - \epsilon_2^2 \sin^2\theta_2}} \right|^2, \qquad (17)$$

but in the limit of high frequency and high velocity, eqs. (8) and (17) lead to the same results, our eqs. (13)-(16),[6] so it is perhaps not necessary to seek better approximations than that of this section.

2.2.2 A Better Approximation – Image Method

The method of sec. 2.2.1 ignored the effects of time-dependent polarization charges near the interface ($z = 0$) between the two semi-infinite dielectric media with (relative) dielectric constants $\epsilon_1(z < 0)$ and $\epsilon_2(z > 0)$. For a better approximation, we recall the image method for dielectrics,[7] that when charge e is at $(0, 0, z)$ in medium 2, the electric field for $z > 0$

[5] See eq. (24.22) of [10], or eq. (2.41) of [8].

[6] Compare with eq. (2.59), p. 35 of [8].

[7] See, for example, sec. 2.1.1 of [12]. Conventions differ in the dielectric image method. Sec. 4.4 of [5] supposes that the image charge is not in vacuum, but in a medium with dielectric constant ϵ_2, while sec. 5.05 of [13] supposes the image charge is in a medium of dielectric constant ϵ_1.

is that in vacuum due to effective charge e/ϵ_2 at $(0,0,z)$ and an image charge $-(e/\epsilon_2)(\epsilon_1 - \epsilon_2)/(\epsilon_1+\epsilon_2)$, while the field for $z<0$ is that due to effective charge $2e/(\epsilon_1+\epsilon_2)$ at $(0,0,z)$ in vacuum.

We consider the case of an observer with $z>0$ (*i.e.*, forward radiation), such that a ray with angle θ_2 to the z-axis at the observer, if it originates with $z<0$, makes angle θ_1 to the z-axis related by Snell's law, $n_1\sin\theta_1 = n_2\sin\theta_2$, *i.e.*, $\sqrt{\epsilon_1}\sin\theta_1 = \sqrt{\epsilon_2}\sin\theta_2$. Then, the frequency-angle spectrum of Čerenkov radiation by charge e with position $\mathbf{x} = vt\,\hat{\mathbf{z}}$ and $v > c/n_{1,2}$ follows from eq. (1) as,[8]

$$\begin{aligned}
\frac{dU_\omega}{d\Omega} &= \frac{\omega^2 n_2}{4\pi^2 c}\left|\int_0^\infty \frac{e}{\epsilon_2}\boldsymbol{\beta}\times\hat{\mathbf{k}}_2\, e^{i\omega t(1-n_2\beta\cos\theta_2)}\,dt + \int_0^\infty -\frac{e}{\epsilon_2}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}(-\boldsymbol{\beta})\times\hat{\mathbf{k}}_2\, e^{i\omega t(1+n_2\beta\cos\theta_2)}\,dt \right.\\
&\qquad\left. + \int_{-\infty}^0 \frac{2e}{\epsilon_1+\epsilon_2}\boldsymbol{\beta}\times\hat{\mathbf{k}}_1\, e^{i\omega t(1-n_1\beta\cos\theta_1)}\,dt\right|^2 \\
&= \frac{e^2\omega^2 n_2 v^2}{4\pi^2 c^3}\left|\frac{1}{\epsilon_2}\sin\theta_2\frac{-1}{i\omega(1-n_2\beta\cos\theta_2)} + \frac{1}{\epsilon_2}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\sin\theta_2\frac{-1}{i\omega(1+n_2\beta\cos\theta_2)}\right.\\
&\qquad\left. + \frac{2}{\epsilon_1+\epsilon_2}\sin\theta_1\frac{1}{i\omega(1-n_1\beta\cos\theta_1)}\right|^2 \\
&= \frac{e^2\sqrt{\epsilon_2}v^2\sin^2\theta_2}{4\pi^2c^3\epsilon_2^2(\epsilon_1+\epsilon_2)^2}\left|\frac{2\epsilon_1 + 2\epsilon_2\sqrt{\epsilon_2}\beta\cos\theta_2}{1-\epsilon_2\beta^2\cos^2\theta_2} - \sqrt{\frac{\epsilon_2}{\epsilon_1}}\frac{2\epsilon_2}{1-\sqrt{\epsilon_1}\beta\sqrt{1-(\epsilon_2/\epsilon_1)\sin^2\theta_2}}\right|^2 \\
&= \frac{e^2 v^2\sqrt{\epsilon_2}\sin^2\theta_2}{\pi^2c^3}\left|\frac{\sqrt{\epsilon_1}\left(\epsilon_1+\epsilon_2^{3/2}\beta\cos\theta_2\right)\left(1-\beta\sqrt{\epsilon_1-\epsilon_2\sin^2\theta_2}\right) - \epsilon_2^{3/2}\left(1-\epsilon_2\beta^2\cos^2\theta_2\right)}{\sqrt{\epsilon_1}\epsilon_2(\epsilon_1+\epsilon_2)\left(1-\epsilon_2\beta^2\cos^2\theta_2\right)\left(1-\beta\sqrt{\epsilon_1-\epsilon_2\sin^2\theta_2}\right)}\right|^2 . \qquad (18)
\end{aligned}$$

This does vanish if $\epsilon_1 = \epsilon_2$, but is not quite the same as the Ginzburg-Frank result (17).

References

[1] K.T. McDonald, *Ph501 Lecture 21*, http://kirkmcd.princeton.edu/examples/ph501/ph501lecture21.pdf

[2] K.T. McDonald, *Čerenkov Radiation by a Neutron and by an Electric Dipole* (Apr. 15, 1984), http://kirkmcd.princeton.edu/examples/cerenkov_n.pdf

[3] L.I. Schiff, *Quantum Mechanics*, (McGraw-Hill, 1949), http://kirkmcd.princeton.edu/examples/QM/schiff_qm_49.pdf

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[5] J.D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, 1975), http://kirkmcd.princeton.edu/examples/EM/jackson_ce2_75.pdf

[8] The contribution from the image charge at $z = -vt$ for $t > 0$, when charge e is at $z = vt$, does not actually originate at $z < 0$, but rather at $z = 0$. As such, for radiation observed at $z > 0$ we use $\mathbf{k}_2$ rather than $\mathbf{k}_1$ in the second integral of the first line of eq. (18).

[6] H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4th ed. (Macmillan, 1961), http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf

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[12] K.T. McDonald, *Dielectric (and Magnetic) Image Methods* (Nov. 21, 2009), http://kirkmcd.princeton.edu/examples/image.pdf

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