Classical Radiation Processes in the Weizsäcker-Williams Approximation

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Abstract

The main features of radiation by relativistic electrons are well approximated in the Weizsäcker-Williams method of virtual quanta. This method is best known for its application to radiation during elementary particle collisions, but is equally useful in describing "classical" radiation emitted during the interaction of a single relativistic electron with an extended system, such as synchrotron radiation, undulator radiation, transition radiation and Čerenkov radiation.

1 The Weizsäcker-Williams Approximation

Following an earlier discussion by Fermi [1], Weizsäcker [2] and Williams [3] noted that the electromagnetic fields of an electron in uniform relativistic motion are predominantly transverse, with $\mathbf{E} \approx \mathbf{B}$ (in Gaussian units). This is very much like the fields of a plane wave, so one is led to regard a fast electron as carrying with it a cloud of virtual photons that it can shed (radiate) if perturbed.

The key feature of the frequency spectrum of the fields can be estimated as follows. To an observer at rest at distance b from the electron's trajectory, the peak electric field is $E = \gamma e/b^2$, and the field remains above half this strength for time $b/\gamma c$, so the frequency spectrum of this pulse extends up to $\omega_{\max} \approx \gamma c/b$. The total energy of the pulse (relevant to this observer) is $U \approx E^2\mathrm{Vol} \approx \gamma^2 e^2/b^4 \cdot b^2 \cdot b/\gamma \approx \gamma e^2/b$.

If the electron radiates all of this energy, the energy spectrum would be,

$$\frac{dU(\omega)}{d\omega} \approx \frac{U}{\omega_{\max}} \approx \frac{e^2}{c}. \tag{1}$$

This result does not depend on the choice of impact parameter b, and is indeed of general validity (to within a factor of $\ln\gamma$). The number n_ω of radiated photons of frequency ω is thus,

$$dn_\omega = \frac{dU(\omega)}{\hbar\omega} \approx \frac{e^2}{\hbar c}\frac{d\omega}{\omega} = \alpha\frac{d\omega}{\omega}, \tag{2}$$

where $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant.

The quick approximation (1)-(2) is not accurate at high frequencies. In general, additional physical arguments are needed to identify the maximum frequency of its validity, called the characteristic or critical frequency ω_C, or equivalently, the minimum relevant impact parameter $b_{\rm min}$. A more detailed evaluation of the high-frequency tail of the virtual photon spectrum shows it to be [1, 2, 3, 4],

$$dn_\omega \approx \alpha \frac{d\omega}{\omega} e^{-2\omega b_{\rm min}/\gamma c} \qquad \text{(high frequency)}. \tag{3}$$

From this, we see the general relation between the critical frequency and the minimum impact parameter is,

$$\omega_C \approx \gamma \frac{c}{b_{\rm min}}, \qquad b_{\rm min} \approx \gamma \lambda_C. \tag{4}$$

The characteristic angular spread θ_C of the radiation pattern near the critical frequency can be estimated from eq. (4) by noting thatthe radiation is much like that of a beam of light with waist $b_{\rm min}$. Then, from the laws of diffraction we conclude that,

$$\theta_C \approx \frac{\lambda_C}{b_{\rm min}} \approx \frac{1}{\gamma}. \tag{5}$$

This behavior is also expected in that a ray of light emitted in the electron's rest frame at 90° appears at angle $1/\gamma$ to the laboratory direction of the electron.

1.1 The Formation Length

To complete an application of the Weizsäcker-Williams method, we must also know over what interval the virtual photon cloud is shaken off the electron to become the radiation detected in the laboratory. Intense (and hence, physically interesting) radiation processes are those in which the entire cloud of virtual photons is emitted as rapidly as possible. This is usefully described by the so-called formation time t_0 and the corresponding formation length $L_0 = vt_0$ where $v \approx c$ is the velocity of the relativistic electron.

The formation length (time) is the distance (time) the electron travels while a radiated wave advances one wavelength λ ahead of the projection of the electron's motion onto the direction of observation. The wave takes on the character of radiation that is no longer tied to its source only after the formation time has elapsed. That is,

$$\lambda = ct_0 - vt_0 \cos\theta \approx L_0(1 - \beta\cos\theta) \approx L_0 \left(\frac{1}{2\gamma^2} + \frac{\theta^2}{2} \right), \tag{6}$$

for radiation observed at angle θ to the electron's trajectory. Thus, the formation length is given by,

$$L_0 \approx \frac{2\lambda}{\theta^2 + 1/\gamma^2} \tag{7}$$

If the frequency of the radiation is near the critical frequency (4), then the radiated intensity is significant only for $\theta \lesssim \theta_C \approx 1/\gamma$, and the formation length is,

$$L_0 \approx \gamma^2 \lambda \qquad (\lambda \approx \lambda_C). \tag{8}$$

A good discussion of the formation length in both classical and quantum contexts has been given in ref. [5].

1.2 Summary of the Method

A relativistic electron carries with it a virtual photon spectrum of α photons per unit frequency interval. When radiation occurs, for whatever reason, the observed frequency spectrum will closely follow this virtual spectrum. In cases where the driving force for the radiation extends over many formation lengths, the spectrum of radiated photons per unit path length for intense processes is given by expressions (2)-(3), which describe the radiation emitted over one formation length, divided by the formation length (7),

$$\frac{dn_\omega}{dl} \approx \frac{\alpha}{L_0(\omega)} \frac{d\omega}{\omega} \times \begin{cases} 1 & (\omega < \omega_C), \\ e^{-\omega/\omega_C} & (\omega \geq \omega_C). \end{cases} \tag{9}$$

Synchrotron radiation, undulator radiation, transition radiation, and Čerenkov radiation are examples of processes which can be described within the context of classical electromagnetism, but for which the Weizsäcker-Williams approximation is also suitable. Čerenkov radiation and transition radiation are often thought of as rather weak processes, but the Weizsäcker-Williams viewpoint indicates that they are actually as intense as is possible for radiation by a single charge, in the sense that the entire virtual photon cloud is liberated over a formation length.

In this paper, we emphasize a simplified version of the Weizsäcker-Williams method with the goal of illustrating the main qualitative features of various radiation processes. A more detailed analysis can reproduce the complete forms of the classical radiation, as has been demonstrated for synchrotron radiation by Lieu and Axford [6]. The Weizsäcker-Williams method can also be used to characterize the radiation from a single oscillating electric charge [7]. The radiation associated with a current pulse that propagates along a conductor and reflects off its end is much like that of an electric charge that has the same velocity ($\approx c$) as the pulse (even though the velocities of the charges in the conductor are much less than c [8].

2 Synchrotron Radiation

Synchrotron radiation arises when a charge, usually an electron, is deflected by a magnetic field. For a large enough region of uniform magnetic field, the electron's trajectory would be a complete circle. However, synchrotron radiation as described below occurs whenever the magnetic field region is longer than a formation length. The radiation observed when the magnetic field extends for less than a formation length has been discussed in [6, 9, 10].

2.1 The Critical Frequency

An important fact about synchrotron radiation is that the frequency spectrum peaks near the critical frequency, ω_C, which depends on the radius R of curvature of the electron's trajectory, and on the Lorentz factor γ via,

$$\omega_C \approx \gamma^3 \frac{c}{R}. \tag{10}$$

Since $\omega_0 = c/R$ is the angular velocity for particles with velocity near the speed of light, synchrotron radiation occurs at very high harmonics of this fundamental frequency. The wavelength at the critical frequency is then,

$$\lambda_C \approx \frac{R}{\gamma^3}. \tag{11}$$

For completeness, we sketch a well-known argument leading to eq. (10). The characteristic frequency ω_C is the reciprocal of the pulselength of the radiation from a single electron according to an observer at rest in the lab. In the case of motion in a circle, the electron emits a cone of radiation of angular width $\theta = 1/\gamma$ according to eq. (5) that rotates with angular velocity $\omega = c/R$. Light within this cone reaches the fixed observer during time interval $\delta t' = \theta/\omega \approx R/\gamma c$. However, this time interval measures the retarded time t' at the source, not the time t at the observer. Both t and t' are measured in the lab frame, and are related by $t' = t - r/c$ where r is the distance between the source and observer. When the source is heading towards the observer, we have, $\delta r = -v\delta t'$, so $\delta t = \delta t'(1 - v/c) \approx \delta t'/2\gamma^2 \approx R/\gamma^3 c$, from which eq. (10) follows.

2.2 The Formation Length

The formation length L_0 introduced in eq. (7) applies for radiation processes during which the electron moves along a straight line, such as Čerenkov radiation and transition radiation. But, synchrotron radiation occurs when the electron moves in the arc of a circle of radius R. During the formation time, the electron moves by formation angle $\theta_0 = L_0/R$ with respect to the center of the circle. We now reconsider the derivation of the formation time, noting that while the electron moves on the arc $R\theta_0 = vt_0$ of the circle, the radiation moves on the chord $2R\sin(\theta_0/2) \approx R\theta_0 - R\theta_0^3/24$. Hence,

$$\begin{aligned} \lambda &= ct_0 - \text{chord} \approx \frac{cR\theta_0}{v} - R\theta_0 + \frac{R\theta_0^3}{24} \\ &\approx R\theta_0(1-\beta) + \frac{R\theta_0^3}{24} \approx \frac{R\theta_0}{2\gamma^2} + \frac{R\theta_0^3}{24}, \end{aligned} \tag{12}$$

for radiation observed at small angles to the chord.

For wavelengths longer than λ_C, the formation angle grows large compared to the characteristic angle $\theta_C \approx 1/\gamma$, and the first term of eq. (12) can be neglected compared to the second. In this case,

$$\theta_0 \approx \left(\frac{\lambda}{R}\right)^{1/3} \approx \frac{1}{\gamma}\left(\frac{\lambda}{\lambda_C}\right)^{1/3} \qquad (\lambda \gg \lambda_C), \tag{13}$$

and,

$$L_0 \approx R^{2/3}\lambda^{1/3} \approx \gamma^2\lambda_C\left(\frac{\lambda}{\lambda_C}\right)^{1/3} \qquad (\lambda \gg \lambda_C), \tag{14}$$

using eq. (11).

The formation angle $\theta_0(\lambda)$ can also be interpreted as the characteristic angular width of the radiation pattern at this wavelength. A result not deducible from the simplified arguments given above is that for $\lambda \gg \lambda_C$, the angular distribution of synchrotron radiation falls off exponentially: $dU(\lambda)/d\Omega \propto e^{-\theta^2/2\theta_0^2}$. See, for example, sec. 14.6 of [4].

For wavelengths much less than λ_C, the formation length is short, the formation angle is small, and the last term of eq. (12) can be neglected. Then, we find that,

$$\theta_0 \approx \frac{\lambda}{\gamma\lambda_C}, \qquad L_0 \approx \gamma^2\lambda \qquad (\lambda \ll \lambda_C), \tag{15}$$

the same as for motion along a straight line, eq. (8). In this limit, our approximation neglects the curvature of the particle's trajectory, which is an essential aspect of synchrotron radiation, and we cannot expect our analysis to be very accurate. But, for $\lambda \ll \lambda_C$, the rate of radiation is negligible.

Of greater physical interest is the region $\lambda \approx \lambda_C$ where the frequency spectrum begins to be exponentially damped but the rate is still reasonably high. The cubic equation (12) does not yield a simple analytic result in the region. So, we interpolate between the limiting results for θ_0 at large and small wavelengths, eqs. (13) and (15), and estimate that,

$$\theta_0 \approx \frac{1}{\gamma}\sqrt{\frac{\lambda}{\lambda_C}} \qquad (\lambda \approx \lambda_C), \tag{16}$$

which agrees with a more detailed analysis [4]. The corresponding formation length $R\theta_0$ is then,

$$L_0 \approx \gamma^2\sqrt{\lambda\lambda_C} \qquad (\lambda \approx \lambda_C). \tag{17}$$

2.3 Transverse Coherence Length

The longitudinal origin of radiation is uncertain to within one formation length L_0. Over this length, the trajectory of the electron is curved, so there is an uncertainty in the transverse origin of the radiation as well. A measure of the transverse uncertainty is the sagitta $L_0^2/8R$, which we label w_0 anticipating a useful analogy with the common notation for the waist of a focused laser beam. For $\lambda \gg \lambda_C$, we have from, eq. (14),

$$w_0 \approx \frac{L_0^2}{R} \approx R^{1/3}\lambda^{2/3} \approx \gamma\lambda_C\left(\frac{\lambda}{\lambda_C}\right)^{2/3} \qquad (\lambda \gg \lambda_C). \tag{18}$$

The sagitta (18) is larger than the minimum transverse length (4), so we expect that the full virtual photon cloud is shaken off over one formation length.

For $\lambda \gg \lambda_C$, the characteristic angular spread (13) of the radiation obeys,

$$\theta_0 \approx \frac{\lambda}{w_0}, \tag{19}$$

consistent with the laws of diffraction. Hence, the distance w_0 of eq. (18) can also be called the transverse coherence length [11] of the source of synchrotron radiation.

The analogy with laser notation is also consistent with identifying the formation length L_0 with the Rayleigh range $z_0 = w_0/\theta_0$, since we see that,

$$L_0 \approx \frac{\lambda}{\theta_0^2} \approx \frac{w_0}{\theta_0}. \tag{20}$$

A subtle difference between the radiation of a relativistic charge and a focused laser beam is that the laser beam has a Gouy phase shift [12, 13] between its waist and the far field, while radiation from a charge does not.

For $\lambda \approx \lambda_C$, the sagitta is $L_0^2/R \approx \gamma^2 \lambda$, using eq. (17). When $\lambda < \lambda_C$, the characteristic angle θ_0 given by eq.(16) is less than λ/sagitta, and the sagitta is no longer a good measure of the transverse coherence length, which is better defined as $\lambda/\theta_0 \approx \gamma\sqrt{\lambda\lambda_C}$.

2.4 Frequency Spectrum

The number of photons radiated per unit path length l during synchrotron radiation is obtained from the Weizsäcker-Williams spectrum (9) using eqs. (14) and (17) for the formation length,

$$\frac{dn_\omega}{dl} \approx \begin{cases} \alpha\omega_C^{2/3} d\omega/\gamma^2 c\omega^{2/3} & (\lambda \gg \lambda_C), \\ \alpha\omega_C^{1/2} e^{-\omega/\omega_C} d\omega/\gamma^2 c\omega^{1/2} & (\lambda \lesssim \lambda_C). \end{cases} \tag{21}$$

We multiply by $\hbar\omega$ to recover the energy spectrum,

$$\frac{dU(\omega)}{dl} \approx \begin{cases} e^2\omega_C^{2/3}\omega^{1/3} d\omega/\gamma^2 c^2 & (\lambda \gg \lambda_C), \\ e^2\omega_C^{1/2}\omega^{1/2} e^{-\omega/\omega_C} d\omega/\gamma^2 c^2 & (\lambda \lesssim \lambda_C). \end{cases} \tag{22}$$

Thus, the Weizsäcker-Williams method shows that the energy spectrum varies as $\omega^{1/3}$ at low frequencies, and as $\sqrt{\omega}e^{-\omega/\omega_C}$ at frequencies above the critical frequency $\omega_C = \gamma^3 c/R$.

The total radiated power is estimated from eq. (22) using $\omega \approx d\omega \approx \omega_C \approx \gamma^3 c/R$, and multiplying by $v \approx c$ to convert dl to dt,

$$\frac{dU}{dt} \approx \frac{e^2\gamma^4 c}{R^2}. \tag{23}$$

This well-known result is also obtained from the Larmor formula, $dU/dt = 2e^2 a^{\star 2}/3c^2$, where the rest-frame acceleration is given by, $a^\star = \gamma^2 a \approx \gamma^2 c^2/R$ in terms of lab quantities.

3 Undulator Radiation

An undulator is a device that creates a region of transverse magnetic field that whose magnitude oscillates with spatial period λ_0 [14, 15, 16]. This field is constant in time, and is usually lies in a transverse plane (although helical undulators have been built, and are actually somewhat easily to analyze). As an electron with velocity v traverses the undulator, its trajectory involves transverse oscillations with laboratory wavelength λ_0, and laboratory frequency $\omega_0 = c/\lambda\!\!\!^{-}_0$. The oscillating electron then emits undulator radiation.

This radiation is usefully described by first transforming to the average rest frame of the electron, which is done by a Lorentz boost of $\gamma = 1/\sqrt{1-(v/c)^2}$ in the first approximation. The undulator wavelength in this frame is $\lambda^\star = \lambda_0/\gamma$, and the frequency of the oscillator is $\omega^\star = \gamma\omega_0$. The electron emits dipole radiation at this frequency in its average rest frame. The laboratory radiation is the transform of this radiation.

Thus, undulator radiation is readily discussed as the Lorentz transform of a Hertzian dipole oscillator, and the Weizsäcker-Williams approximation does not offer much practical advantage here.[1] However, an analysis of undulator radiation can validate the Weizsäcker-Williams approximation, while also exploring the distinction between undulator radiation and wiggler radiation.

3.1 A First Estimate

The formation length, defined as the distance over which radiation pulls one wavelength ahead of the electron, is $L_0 \approx \gamma^2\lambda \approx \lambda_0$, the undulator period. Hence, the radiation from different periods of the undulator does not interfere, such that the intensity of the radiation is proportional to the number of periods. Also, the frequency spectrum of the radiation in independent of the number of periods

The characteristic angle of undulator radiation in the laboratory is $\theta_C \approx 1/\gamma$, this being the transform of a ray at $\theta^\star = 90^\circ$ to the electron's lab velocity. The frequency of the radiation at this angle is,

$$\omega_C \approx \gamma\omega^\star = \gamma^2\omega_0, \tag{24}$$

and wavelength

$$\lambda_C \approx \frac{\lambda_0}{\gamma^2}. \tag{25}$$

Radiation with maximum frequency is emitted at $\theta = 0$, at which,

$$\omega_{\max} \approx 2\gamma\omega^\star = 2\gamma^2\omega_0. \tag{26}$$

Hence, the bandwidth of undulator radiation is

$$\frac{\Delta\omega}{\omega_C} \approx 1. \tag{27}$$

According to the Weizsäcker-Williams approximation (9), if the undulator field is strong enough to cause the electron to shed its cloud of virtual photons, then,

$$\frac{dn_\omega}{dl} \approx \frac{\alpha}{L_0}\frac{d\omega}{\omega} \approx \frac{\alpha}{\lambda_0}, \tag{28}$$

so the frequency spectrum is roughly flat up to $\omega_{\max}$. The radiated power is $v\hbar\omega_C \approx c\hbar\omega_C$ times eq. (28),

$$\frac{dU}{dt} \approx \frac{e^2 c\gamma^2}{\lambda_0^2}, \tag{29}$$

using eq. (24).

To clarify the notion of the strength of an undulator we need to examine the electron's trajectory through the undulator in greater detail.

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[1] However, even Hertzian dipole radiation can be treated by the Weizsäcker-Williams approximation [7].

3.2 Details of the Electron's Trajectory

A magnetic field changes the direction of the electron's velocity, but not its magnitude. As a result of the transverse oscillation in the undulator, the electron's average forward velocity $\overline{v}$ will be less than v. The boost to the average rest frame is described by $\overline{\gamma}$ rather than γ.

In the average rest frame, the electron is not at rest, but oscillates in the electric and magnetic fields $\tilde{E} \approx \tilde{B} = \overline{\gamma} B_0$, where we use the symbol $\tilde{}$ to indicate quantities in the average rest frame. The case of a helical undulator is actually simpler than that of a linear one. For a helical undulator, the average-rest-frame fields are essentially those of circularly polarized light of frequency $\tilde{\omega} = \overline{\gamma}\omega_0$. The electron moves in a circle of radius R at this frequency, in phase with the electric field $\tilde{E}$, and with velocity $\tilde{v}$ and associated Lorentz factor $\tilde{\gamma}$, all related by,

$$\frac{\tilde{\gamma} m \tilde{v}^2}{R} = \tilde{\gamma} m \tilde{v} \tilde{\omega} = e\tilde{E}. \tag{30}$$

From this we learn that,

$$\tilde{\gamma}\tilde{\beta} = \frac{e\tilde{E}}{m\tilde{\omega}c} \approx \frac{eB_0}{m\omega_0 c} \equiv \eta, \tag{31}$$

and hence,

$$\tilde{\gamma} = \sqrt{1+\eta^2}, \qquad \tilde{\beta} = \frac{\eta}{\sqrt{1+\eta^2}}, \tag{32}$$

and,

$$R = \frac{\tilde{\beta}c}{\tilde{\omega}} = \frac{\eta \tilde{\lambda\!\!\!^{-}}}{\sqrt{1+\eta^2}} = \frac{\eta \lambda\!\!\!^{-}_0}{\overline{\gamma}\sqrt{1+\eta^2}} \tag{33}$$

Thus, the dimensionless parameter η describes many features of the transverse motion of an electron in an oscillatory field. It is a Lorentz invariant, being proportional to the magnitude of the 4-vector potential.

For a linear undulator, η is usefully defined as,

$$\eta = \frac{eB_{0,\mathrm{rms}}}{m\omega_0 c}, \tag{34}$$

where the root-mean-square (rms) average is taken over one period. With the definition (34), the rms values of $\tilde{\beta}$, $\tilde{\gamma}$ and R for a linear undulator of strength η are also given by eqs. (32)-(33).

We can now display a relation for $\overline{\gamma}$ by noting that in the average rest frame, the electron's (average) energy is $\tilde{\gamma} mc^2 = m\sqrt{1+\eta^2}c^2$, while its average momentum is zero there. Hence, on transforming back to the lab frame, we have $\gamma mc^2 = \overline{\gamma}\tilde{\gamma} mc^2$, and so,

$$\overline{\gamma} = \frac{\gamma}{\sqrt{1+\eta^2}}. \tag{35}$$

The maximum frequency of the (first-harmonic) undulator radiation is,

$$\omega_{\max} = 2\overline{\gamma}^2\omega_0 = \frac{2\gamma^2\omega_0}{1+\eta^2}\,. \tag{36}$$

The transverse amplitude of the motion is obtained from eqs. (33) and (35),

$$R = \frac{\eta \lambda\!\!\!^{-}_0}{\gamma} = 2\eta\gamma \lambda\!\!\!^{-}_C, \tag{37}$$

recalling eq. (25).

3.3 $\eta < 1$: Weak Undulators

The estimate (29) for the power of undulator radiation holds only if essentially the whole virtual photon cloud around the electron is shaken off. This can be expected to happen only if the amplitude of the electron's transverse motion exceeds the minimum impact parameter $b_{\rm min} \approx \gamma\lambda_C$ introduced in eq. (4). From eq. (37) we see that the transverse amplitude obeys,

$$R \approx \eta b_{\rm min}. \tag{38}$$

Thus, for η less than one, the undulator radiation will be less than full strength. We readily expect that the intensity of weak radiation varies as the square of the amplitude of the motion, so the estimate (29) should be revised as,

$$\frac{dU}{dt} \approx \frac{\eta^2 e^2 c \gamma^2}{\lambda\!\!\!^{-}{}_0^2}, \qquad (\eta \lesssim 1). \tag{39}$$

The radiated power can be calculated exactly via the Larmor formula,

$$\frac{dU}{dt} = \frac{2e^2 a^{\star 2}}{3c^3}, \tag{40}$$

where $a^\star = eE^\star/m$ is the acceleration of the electron in its instantaneous rest frame. The electron is moving in a helix with its velocity perpendicular to $\mathbf{B}_0$, so the electric field in the instantaneous rest frame is $E^\star = \gamma\beta B_0 \approx \gamma B_0$. Hence,

$$\frac{dU}{dt} \approx \frac{2e^2\gamma^2}{3c}\left(\frac{eB_0}{mc}\right)^2 = \frac{2e^2 c\gamma^2\eta^2}{3\lambda\!\!\!^{-}{}_0^2}, \tag{41}$$

in agreement with the revised estimate (39).

In practice, $\eta \approx 1$ is the region of greatest interest as it provides the maximum amount of radiation at the fundamental frequency ω_C.

3.4 $\eta > 1$: Wiggler Radiation

For $\eta \gtrsim 1$ the motion of the electron in its average rest frame is relativistic, as can be see from eq. (32). In this case, higher-multipole radiation becomes important, which appears at integer multiples of frequency $\omega^\star$ in the average rest frame, and at the corresponding Lorentz transformed frequencies in the lab frame. The total radiated power is still given by eq. (29), so the amount of power radiated any particular frequency is less than when $\eta \approx 1$.

For $\eta \gg 1$ the motion of the electron is ultrarelativistic in the average rest frame, with $\tilde{\gamma} \approx \eta$, according to eq. (32). The critical frequency $\omega_C^\star$ of the radiation in the average rest frame is,

$$\omega_C^\star \approx \tilde{\gamma}^3 \omega^\star \approx \eta^3 \overline{\gamma} \omega_0 \approx \eta^2 \gamma \omega^0, \tag{42}$$

since $\overline{\gamma} \approx \gamma/\eta$ for $\gamma \gg \eta \gg 1$, recalling eq. (35). In the laboratory, the critical frequency is then,

$$\omega_C \approx \overline{\gamma} \omega_C^\star \approx \tilde{\gamma}^3 \omega^\star \approx \eta^3 \overline{\gamma} \omega_0 \approx \eta \gamma^2 \omega^0, \tag{43}$$

i.e., about η times the critical frequency $\gamma^2 \omega_0$ for undulator radiation ($\eta \lesssim 1$).

4 Transition Radiation

As a charged particle crosses, for example, a vacuum/metal boundary, its interaction with charges in the material results in their acceleration and hence radiation, commonly called transition radiation. The formation zone extends outwards from each boundary, with formation length given by eq. (7). The number of photons emitted as the particle crosses each boundary is given by eq. (2) as α per unit frequency interval. If two boundaries are separated by less than a formation length, interference effects arise that will not be considered here.

The minimum relevant transverse scale, $b_{\rm min}$, is the plasma wavelength $\lambda\!\!\!^{-}_p = c/\omega_p$, so the critical frequency is $\omega_C \approx \gamma \omega_p$, according to eq. (4). This is well into the x-ray regime. While the characteristic angle of transition radiation is $1/\gamma$, there is only a power-law falloff at larger angles, and the optical transition radiation from an intense beam of charged particles can be used to measure the spot size to accuracy of a few λ [17, 18].

5 Čerenkov Radiation

When a charged particle moves with velocity v in a dielectric medium, Čerenkov radiation is emitted at those wavelengths for which $v > c/n$, where $n(\lambda)$ is the index of refraction. As the particle approaches lightspeed in a medium, its electric field is compressed into a "pancake", which deforms into a cone when the the particle velocity exceeds lightspeed. The particle outruns its electric field, which is freed as the Čerenkov radiation. The fields become identifiable as radiation after the particle has moved a formation length, $L_0 = vt_0$, which is the distance over which the electron pulls one wavelength ahead of the projection of the wave motion onto the electron's direction. The Čerenkov angle θ_C is defined by the direction of the radiation, which is normal to the conical surface that contains the electric field. As usual, $\cos\theta_C = c/nv = 1/n\beta$. The formation length is then,

$$\lambda = vt_0 - \frac{c}{n} t_0 \cos\theta_C = L_0 \sin^2\theta_C. \tag{44}$$

Thus $L_0 = \lambda / \sin^2\theta_C$, and the photon spectrum per unit path length from eq. (2) is,

$$\frac{dn_\omega}{dl} \approx \frac{\alpha}{L_0} \frac{d\omega}{\omega} \approx \frac{\alpha \sin^2\theta_C}{\lambda} \frac{d\omega}{\omega} \approx \alpha \sin^2\theta_C \frac{d\omega}{c}, \tag{45}$$

as is well-known.

The characteristic angle θ_C of Čerenkov radiation is essentially independent of the Lorentz factor γ of the charged particle, unlike that for the other radiation processes considered here. Correspondingly, the characteristic transverse length b associated with Čerenkov radiation is also largely independent of γ. Rather, the region over which the Čerenkov radiation develops has radius roughly that of the Čerenkov cone after one formation length, *i.e.*, $b = L_0 \cos\theta_C \sin\theta_C \approx \lambda / \tan\theta_C$. This is large only near the Čerenkov threshold where the radiated intensity is very small.

That the formation radius for Čerenkov radiation is of order λ is supported by an analysis [19] of a particle moving in vacuum along the axis of a tube inside a dielectric medium; the calculated Čerenkov radiation is negligible at wavelengths larger than the radius of the tube.

Čerenkov radiation is a form of energy loss for a particle passing through a medium, and is related to so-called ionization loss (see, for example, secs. 13.1-4 of [4]). The latter is important for frequencies higher than the ionization potential (divided by $\hbar$) of the medium, for which the index of refraction is typically less than one. At frequencies that can cause ionization, the Čerenkov effect is insignificant. The transverse scale of the ionizing fields grows with γ due to relativistic flattening, but shielding due to the induced dielectric polarization of the medium results in an effective transverse scale $b \propto \sqrt{\gamma}$ for these fields when $\gamma \gg 1$.

References

[1] E. Fermi, *Über die Theorie des Stoßes zwischen Atomen und elektrisch geladenen Teilchen*, Z. Phys. **29**, 315 (1924), http://kirkmcd.princeton.edu/examples/EM/fermi_zp_29_315_24.pdf

[2] C.F. von Weizsäcker, *Ausstrahlung bei Stößen sehr schneller Elektronen*, Z. Phys. **88**, 612 (1934), http://kirkmcd.princeton.edu/examples/EM/weizsacker_zp_88_612_34.pdf

[3] E.J. Williams, *Correlation of Certain Collision Problems with Radiation Theory*, Kgl. Danske Videnskab. Selskab Mat.-Fys. Medd. **13**, No. 4 (1935), http://kirkmcd.princeton.edu/examples/QED/williams_dkdvsmfm_13_4_1_35.pdf

[4] J.D. Jackson, *Classical Electrodynamics*, 3$^{\rm rd}$ ed. (Wiley, 1999), http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[5] S. Klein, *Suppression of bremsstrahlung and pair production due to environmental factors*, Rev. Mod. Phys. **75**, 1501 (1999), http://kirkmcd.princeton.edu/examples/accel/klein_rmp_75_1501_99.pdf

[6] R. Lieu and W.I. Axford, *Synchrotron Radiation: an Inverse Compton Effect*, Ap. J. **416**, 700 (1993), http://kirkmcd.princeton.edu/examples/accel/lieu_apj_416_700_93.pdf *Quantum-Limited Synchrotron Radiation in Inhomogeneous Magnetic Fields*, *ibid.* **447**, 302 (1995), http://kirkmcd.princeton.edu/examples/accel/lieu_apj_447_302_95.pdf

[7] K.T. McDonald and M.S. Zolotorev, *Hertzian Dipole Radiation via the Weizsäcker-Williams Method* (Aug. 4, 2003), http://kirkmcd.princeton.edu/examples/hertzian.pdf

[8] T.W. Hertel and G.S. Smith, *Pulse Radiation from an Insulated Antenna: An Analog of Cherenkov Radiation from a Moving Charged Particle*, IEEE Trans. Ant. Prop. **48**, 165 (2000), http://kirkmcd.princeton.edu/examples/EM/hertel_ieeetap_48_165_00.pdf

[9] R. Coïsson, *On Synchrotron Radiation in Short Magnets*, Opt. Comm. **22**, 135 (1977), http://kirkmcd.princeton.edu/examples/accel/coisson_oc_22_135_77.pdf

[10] V.G. Bagrov *et al.*, *Radiation of Relativistic Electrons Moving in an Arc of a Circle*, Phys. Rev. D **28**, 2464 (1983), http://kirkmcd.princeton.edu/examples/accel/bagrov_prd_28_2464_83.pdf

[11] See, for example, chap. 4 of L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge U. Press, 1995).

[12] G. Gouy, *Sur une propriété nouvelle des ondes lumineuse*, Compt. Rendue (Paris) **110**, 1251 (1890), http://kirkmcd.princeton.edu/examples/optics/gouy_cr_110_1251_90.pdf
Sur la propagation anomele des ondes, *ibid.* **111**, 33 (1890), http://kirkmcd.princeton.edu/examples/optics/gouy_cr_111_33_90.pdf
Ann. Chim. Phys. **24**, 145 (1891), http://kirkmcd.princeton.edu/examples/optics/gouy_acr_124_145_91.pdf
See also sec. 17.4 of [13].

[13] A.E. Siegman, *Lasers* (University Science Books, Mill Valley, CA, 1986), http://kirkmcd.princeton.edu/examples/EM/siegman_86.pdf

[14] R.C. Wingerson, *"Corkscrew" – A Device for Changing the Magnetic Moment of Charged Particles in a Magnetic Field*, Phys. Rev. Lett. **6** 446 (1961), http://kirkmcd.princeton.edu/examples/accel/wingerson_prl_6_446_61.pdf

[15] J.P. Blewett and R. Chasman, *Orbits and fields in the helical wiggler*, J. Appl. Phys. **48**, 2692 (1977), http://kirkmcd.princeton.edu/examples/accel/blewett_jap_48_2692_77.pdf

[16] B.M. Kincaid, *A short-period helical wiggler as an improved source of synchrotron radiation*, J. Appl. Phys. **48**, 2684 (1977), http://kirkmcd.princeton.edu/examples/accel/kincaid_jap_48_2684_77.pdf

[17] X. Artru *et al.*, *Experimental investigations on geometrical resolution of optical transition radiation (OTR)*, Nucl. Instr. and Meth. A **410**, 148 (1998), http://kirkmcd.princeton.edu/examples/accel/artru_nim_a410_148_98.pdf *Resolution power of optical transition radiation: Theoretical considerations*, *ibid.* B **145**, 160 (1998), http://kirkmcd.princeton.edu/examples/accel/artru_nim_b145_160_98.pdf

[18] P. Catravas *et al.*, *Beam Profile Measurement at 30 GeV Using Optical Transition Radiation*, PAC99, WEA100, http://accelconf.web.cern.ch/accelconf/p99/PAPERS/WEA100.PDF
http://kirkmcd.princeton.edu/examples/accel/catravas_pac99_2111.pdf

[19] V.L. Ginzburg and I.M. Frank, *Radiation of Electrons and Atoms Moving Along the Axis of a Tube in a Dense Medium*, Dokl. Akad. Nauk, SSSR **56**, 699 (1947); summarized in sec. 3.4 of J.V. Jelley, *Čerenkov Radiation and Its Applications* (Pergamon Press, 1958), http://kirkmcd.princeton.edu/examples/detectors/jelley_cerenkov_radiation.pdf