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Compression of Beam Energy Via Off-Axis Traversal of an RF Cavity

A \$100 Misunderstanding?

1 Introduction

As a possible step in the cooling of the longitudinal phase space of the muon beams at a muon collider, we consider the use of a simple RF cavity to compress the momentum, as shown in Fig. 1. The cavity would be located at a point where the beam is dispersed in momentum, *i.e.*, where that momentum varies linearly with position along a transverse axis. The beam would enter the cavity (passing through the wall!) near the outer edge of the cavity where the electric field varies linearly with position. If the beam passes through the cavity when the field is maximum the energy gain is a linear function of transverse position. In principle a linear dispersion of the beam can be completely cancelled by this procedure.

The finite extent of the beam along its axis results in less than perfect compensation of the dispersion. It also results in some growth of angles of beam particles relative to the transverse axis of dispersion. This interchange of longitudinal for transverse phase volume could be compensated by further transverse cooling of the beam.

R.B. Palmer notes that he once lost a \$100 bet with E. Courant regarding schemes for manipulation of longitudinal phase space. It remains to be seen whether the present proposal is among the disfavored class.[1]

2 Details

2.1 Energy Compression

We suppose that the beam has been dispersed so that at location $z = 0$ along its path the momentum p varies as

$$p = p_0 + k(x - \bar{x}), \tag{1}$$

where p_0 is the central momentum, the x-axis is the transverse axis of dispersion and $\bar{x}$ is the centroid of the beam at $z = 0$.

[1] A paper by Courant can be interpreted as casting doubt on whether the "rf wedge" can work.
http://puhep1.princeton.edu/~mcdonald/examples/accel/courant_pmp_257_66.pdf
We believe that Courant's theorem does not apply because the energy increments provided by an rf cavity cannot be described by a "linear" transformation in Courant's sense.

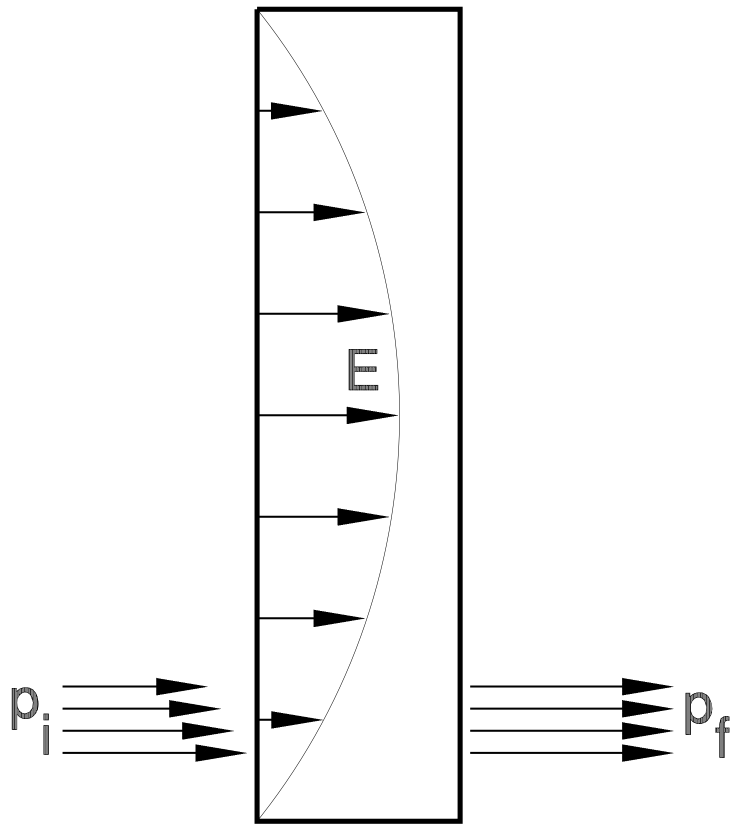

Figure 1: The muon beam passes through the RF cavity wall at a momentum dispersion point that is well off the cavity axis.

An rf cavity is centered on $(x, y, z) = (0, 0, 0)$. To simplify the calculations we suppose the cavity is a rectangular box of extent a in x and y and length b along z. The cavity is excited in the $\mathrm{TE}_{1,1,0}$ mode for which the fields are (in cgs units)

$$\begin{aligned}
E_x = E_y &= 0, \\
E_z &= E_0 \cos\frac{\pi x}{a} \cos\frac{\pi y}{a} \cos\omega t, \\
B_x &= \frac{c}{\omega}\frac{\pi}{a} E_0 \cos\frac{\pi x}{a} \sin\frac{\pi y}{a} \sin\omega t, \\
B_y &= \frac{c}{\omega}\frac{\pi}{a} E_0 \sin\frac{\pi x}{a} \cos\frac{\pi y}{a} \sin\omega t, \\
B_z &= 0,
\end{aligned} \tag{2}$$

where E_0 is the peak electric field, ω is the angular frequency and c is the speed of light.

We make the impulse approximation that the cavity fields affect the energy and momenta of the beam particles but not their trajectories while they are within the cavity. The trajectory of a typical beam particle can then be parametrized (within the cavity) as

$$x = x_0 + \beta_x c t,$$

$$
\begin{aligned}
y &= y_0 + \beta_y ct, \\
z &= z_0 + \beta_z ct.
\end{aligned} \tag{3}
$$

The particle is within the cavity during the interval

$$
[t_{min}, t_{max}] = \left[-\frac{b}{2\beta_z c} - \frac{z_0}{\beta_z c}, \frac{b}{2\beta_z c} - \frac{z_0}{\beta_z c} \right]. \tag{4}
$$

The rate of energy gain on traversing the cavity is $dU/dt = e\mathbf{E} \cdot \mathbf{v}$ where e and $\mathbf{v}$ are the charge and velocity of the particle. The energy gain is then

$$
\Delta U = e\beta_z c \int_{t_{min}}^{t_{max}} E_z dt = e\beta_z c E_0 \int_{t_{min}}^{t_{max}} \cos\frac{\pi(x_0 + \beta_x ct)}{a} \cos\frac{\pi(y_0 + \beta_y ct)}{a} \cos\omega t \, dt. \tag{5}
$$

We evaluate this integral in the approximation that $y_0 \ll a$ and $z_0 \ll b$ but that x_0 is near the outer edge of the cavity with $a/2 - x_0 \ll a/2$. We also suppose that the frequency of the cavity is low enough that $\omega t \ll 1$ while the particles are in the cavity. Finally, we suppose that the beam is paraxial with $\beta_x \ll \beta_z$ and $\beta_y \ll \beta_z$. We expand the sines and cosines and keep terms to first order. In particular, $\sin(\pi x_0/a) \approx 1$ while $\cos(\pi x_0/a) \approx \pi(a/2 - x_0)/a$. Then

$$
\Delta U \approx \frac{\pi e\beta_z c E_0}{a} \int_{t_{min}}^{t_{max}} (a/2 - x_0 - \beta_x ct) \cos\omega t \, dt \approx \pi e E_0 \frac{b}{a} \left(\frac{a}{2} - x_0 + z_0 \frac{\beta_x}{\beta_z} \right). \tag{6}
$$

The third term in eq. (6) is of higher order than the first two.

The effect of the cavity on a particle of initial energy U_i is obtained after rewriting eq. (1) as

$$
\begin{aligned}
U_i &= \sqrt{p^2c^2 + m^2c^4} = \sqrt{[p_0 + k(x_0 - \bar{x})]^2 c^2 + m^2c^4} \\
&\approx U_0 \sqrt{1 + 2k(x_0 - \bar{x}) \frac{p_0 c^2}{U_0^2}} \approx U_0 + k\beta_0 c(x_0 - \bar{x}),
\end{aligned} \tag{7}
$$

where $U_0 = \sqrt{p_0^2 c^2 + m^2 c^4}$ is the central energy and $\beta_0 = p_0 c/U_0$ and we suppose that the initial momentum spread $\Delta p/p_0$ is small. Combining eqs. (6) and (7), the final energy is

$$
U_f = U_0 + k\beta_0 c(x_0 - \bar{x}) + \pi e E_0 \frac{b}{a} \left(\frac{a}{2} - x_0 + z_0 \frac{\beta_x}{\beta_z} \right). \tag{8}
$$

The cavity field E_0 should be chosen to satisfy

$$
E_0 = \frac{a}{b} \frac{k\beta_0 c}{\pi e}. \tag{9}
$$

In this case the final energy is

$$
U_f = U_i + \Delta U = U_0 + \pi e E_0 \frac{b}{a} \left(\frac{a}{2} - \bar{x} + z_0 \frac{\beta_x}{\beta_z} \right). \tag{10}
$$

The final energy is constant except for the term in z_0, the longitudinal coordinate of the particle relative to the center of the bunch.

The effect of the term in z_0 is not necessarily bad. If the momentum dispersion is obtained in, say, a 180° bending magnet the higher momentum particles travel longer paths and will arrive at the rf cavity later than lower momentum particles. Hence $z_0 = -k'p$ for some positive constant k'. If in addition there were a positive correlation of the average x-angle, $\bar{\beta}_x/\bar{\beta}_z$, with energy then the energy kick in z_0 would compress the beam in z as well.

2.2 Transverse Kicks

For particles with $z_0 \neq 0$ the cavity causes a small kick in the x-component of transverse momentum. To see this we calculate

$$\begin{aligned}
\Delta p_x &= \int_{t_{min}}^{t_{max}} F_x dt = -e\beta_z c \int_{t_{min}}^{t_{max}} B_y dt \\
&= -\frac{\pi c}{a\omega} e\beta_z c \int_{t_{min}}^{t_{max}} \sin\frac{\pi(x_0+\beta_x ct)}{a} \cos\frac{\pi(y_0+\beta_y ct)}{a} \sin\omega t\, dt \\
&\approx -\frac{\pi c}{a\omega} e\beta_z c \int_{t_{min}}^{t_{max}} \sin\omega t\, dt \approx -\pi e E_0 \frac{b}{a}\frac{z_0}{\beta_z c},
\end{aligned} \tag{11}$$

using the Lorentz force law and eq. (3). Comparing with eq. (6) we see that $\Delta p_z = \Delta U/\beta_z c$ and

$$\frac{\Delta p_x}{\Delta p_z} \approx \frac{z_0}{a/2 - x_0}. \tag{12}$$

This could be troublesome if z_0 is of the same order as $a/2 - x_0$. It appears desirable to use a cavity large enough that the bunch length ($\approx z_0$) is small compared to the distance $a/2 - \bar{x}$ of the beam from the outer edge of the cavity. Of course, we should maintain the condition $a/2 - \bar{x} \ll a/2$ for the preceding analysis to hold.

The kick in the y-direction is negligible:

$$\begin{aligned}
\Delta p_y &= \int_{t_{min}}^{t_{max}} F_y dt = e\beta_z c \int_{t_{min}}^{t_{max}} B_x dt \\
&= \frac{\pi c}{a\omega} e\beta_z c \int_{t_{min}}^{t_{max}} \cos\frac{\pi(x_0+\beta_x ct)}{a} \sin\frac{\pi(y_0+\beta_y ct)}{a} \sin\omega t\, dt \approx 0.
\end{aligned} \tag{13}$$