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Comments on Ionization Cooling

1 Introduction

This note began in Oct. 1998 as a commentary on Sec. V of the Status Report [1], but became a forum for a larger commentary on ionization cooling. The present version omits the editorial remarks, and corrects an error in Sec. 5. Section 2 presents a series of general comments on ionization cooling. Section 3 presents a derivation of analytic approximations for transverse and longitudinal ionization cooling. Section 4 discusses the role of canonical angular momentum in solenoid transport. Section 5 discusses the applications of the concept of betatron oscillations to solenoids.

Shortly after these notes were written, related notes by Palmer and Fernow [2] we presented at the Jan. 1999 Vanderbilt Particle Accelerator School.

2 General Comments on Ionization Cooling

- **Where do equations SR(24) and SR(26) come from?**

  These describe the rate of change with position of normalized transverse emittance ϵ_N and of energy spread ΔE for a muon bunch of central energy E and Lorentz factors β and γ traversing an absorber of radiation length L_R while inside a confining channel with local betatron parameter $\beta_\perp$,

  $$\frac{d\epsilon_N}{dz} = \frac{\epsilon_N}{\beta^2 E}\frac{dE}{dz} + \frac{\beta_\perp (13.6\ \mathrm{MeV}/c)^2}{2\beta^3 E m_\mu L_R}. \qquad \mathrm{SR}(24)$$

  $$\frac{d(\Delta E)^2}{dz} = 2(\Delta E)^2 \frac{d}{dE}\frac{dE_0}{dz} + \frac{d(\Delta E)^2_{\mathrm{straggling}}}{dz}. \qquad \mathrm{SR}(26).$$

  [SR means Status Report. Equations without SR in front refer to the present paper. In this note I follow the usual practice in high energy physics that the energy loss dE/dx is negative; in the Status Report that quantity dE/ds is considered to be positive.]

  Something close to these was given by Neuffer in 1983 [3], but Dave seems to draw different conclusions from them than does Bob Palmer.

  A partial version of these equations was given earlier by Skrinsky [4], who also drew inferences from them closer to those of Neuffer than to Palmer.

  These early works assumed that fairly high energy muons were being cooled, and so approximated $E_\mu = P_\mu c$.

The earliest appearance of SR(24) that I have found is in two useful papers by Fernow and Gallardo [5, 6]; however, the derivation there includes the phrase "it can be shown". I believe that SR(24) is meant to improve on earlier versions by being applicable for muons of any β. A result somewhat like SR(24) appears in sec. 5.3 of [7].

Equation SR(26) seems to have first appeared in [8] by Neuffer, and differs from an earlier version by a factor of 2 in the first term.

- **Are equations SR(24) and SR(26) correct?**

  I give a detailed derivation of them in sec. 3 below, which provides some sense of the approximations underlying these equations.

- **Has a numerical study ever been performed with the purpose of verifying equations SR(24) and SR(26)?**

  I think not. This might make a good introductory project for someone who wishes to combine the insights of analytic and numerical calculations.

- **Why don't we try anymore to obtain longitudinal ionization cooling?**

  Both [3, 4] discuss longitudinal cooling before transverse cooling. They note that the dE/dx curve has positive slope for muon momenta above about 400 MeV/c, so longitudinal cooling might be possible here. Hence, they emphasize muon cooling at momenta higher than this.

  My answer to this question is contained in the next comment.

- **Why are we now at momenta lower than 400 MeV/c in the cooling section?**

  One answer is: **money**. It costs more in rf systems to operate at higher energy, since the process of cooling takes the beam energy away and replenishes it about 30 times. But, if we choose a parameter region in which cooling proves to be impossible for physics reasons, the cost savings is illusory. [Plus, I will comment later that cost optimization might well take us far from the present parameter set.]

  I have thought of **another answer**, which I have never heard discussed, although it is a variant of a remark by Palmer. We plan to replenish the lost beam energy with rf cavities, which requires the beam to be bunched. The average velocity of this bunch along the rf axis, z, must obey $\bar{\beta}_z < 1$. But if the bunch has large emittance, some particles will have large angles. Even if these particles had $\beta = 1$, their longitudinal velocity would be only $\beta_z = \cos\theta_z$. To keep these particles in the bunch, we must have $\bar{\beta}_z \le \cos\theta_{z,\mathrm{max}}$.

  A practical fact is needed here: what is a realistic $\theta_{z,\mathrm{max}}$? If we choose 30°, then we immediately find $\cos\theta_{z,\mathrm{max}} = 0.866 = \bar{\beta}_z$, so the central $\gamma = 2$, and the central momentum $\bar{P}_\mu = 182$ MeV/c.

  In sum, **cooling of a beam with large angular spread $\Leftrightarrow$ low central momentum $\Rightarrow$ must give up longitudinal ionization cooling** (of the simplest sort).

  Note that we have also demonstrated the important result that **particles with large angles must have higher momentum to stay within the bunch**.

Furthermore, there must by some kind of transverse confinement of the beam particles, or they would wander off. In general, we expect that larger angle particles will have trajectories with larger transverse amplitude. Hence, **particles with large amplitudes of transverse oscillation must have higher momentum to stay within the bunch**.

- **Why not cool at even lower momentum?** It might be easier to hold a large-emittance bunch together if the highest momentum didn't have to correspond to β very close to 1.

  We first look for an answer from eq. SR(24), from which we learn that there is an equilibrium emittance below which one cannot cool (with a given apparatus), obtained when SR(24) vanishes,

$$\epsilon_{N,\min} = \frac{\beta_\perp (13.6 \text{ MeV}/c)^2}{2\beta m_\mu \langle -dE_\mu/ds \rangle L_R}. \tag{1}$$

  [This result appears in the Status Report as SR(25).] *To be consistent with the notation used in this Comment, I have added the minus sign to dE/ds.*

  The idea of an equilibrium emittance appeared already in [9].

  It is claimed in [10] that the betatron function for a particle of charge e and longitudinal momentum P_z in a solenoid with magnetic field B_z obeys,

$$\beta_{\perp,\text{solenoid}} = \frac{2cP_z}{eB_z}. \tag{2}$$

  See sec. 5 below for discussion of this.

  We learn 3 key things about transverse cooling from this one result,

  1. The radiation length L_R should be long. This is fairly obvious, and was noted by Ado and Balbekov [9]. Liquid hydrogen is best.
  2. The absorber should be placed at a low-$\beta_\perp$ point. This follows also by noting that the "heating" term in SR(24), due to multiple scattering, is less dangerous if the particles' angles are large, which occurs when the $\beta_\perp$ is small.

     This insight can be traced to Skrinsky [11].

     Equation (2) reminds us that low-$\beta_\perp$ requires strong magnetic fields.
  3. $\beta \langle dE_\mu/ds \rangle$ should be as big as possible. For β slightly less than 1, $dE_\mu/ds \propto 1/\beta^{5/3}$. See Figure 23.1 of the particle data tables. Then, in the region of interest to us, $\beta(dE_\mu/ds) \propto 1/\beta^{2/3}$, *i.e.*, $\epsilon_{N,\min} \propto \beta^{2/3}$. Hence, it appears from SR(24) that there might indeed be some advantage in going to lower $\bar{\beta}$.

     If the bunch is transported in a solenoid, eq. (2) indicates that an additional factor of $\bar{\beta}$ appears in the expression for $\epsilon_{N,\min}$, which reinforces the interest in lower $\bar{\beta}$.

  I don't know the answer to the question of the optimum $\bar{\beta}$. Could it actually be a serious question?

– Some guidance comes from eq. SR(26). For low β the first term becomes positive and adds to the undesirable "heating" of straggling. The first term changes sign at $\gamma\beta = 4$ (for hydrogen). This suggests that going lower may aggravate our already difficult problem with longitudinal emittance.

  In sec. 3.2.1, we show that the "heating" of $(\Delta E)^2$ varies as $1/\beta^4$ for low γ, which is probably the most quantitative argument against use of very low β.

  For β too low, particles with lower than average energy would just stop in the absorber. It would be interesting to have a quantitative statement of where this problem sets in.

– The money argument favors lower β.

– Lower β means lower γ, so the muons decay faster in the lab. This appears to favor higher β. But the length of the rf cavities needed to maintain the energy vary as γ (really $\gamma - 1$), so the decay loss is independent of γ to a first approximation.

- **What is $\theta_{x,\max}$?**

  This question occurred in a previous comment, but deserves elaboration. I follow [10] here.

  [I am changing notation here: θ_x is the projection onto the x-z plane of what was called θ_z earlier.]

  Using the relation $\sigma_{\theta_x} = \sqrt{\epsilon/\beta_\perp} = \sqrt{\epsilon_N/\gamma\beta\beta_\perp}$, our eq. (1) becomes,

$$\sigma^2_{\theta_{x,\min}} = \frac{(13.6 \text{ MeV})^2}{2\beta E\langle -dE_\mu/ds\rangle L_R}. \tag{3}$$

  Suppose we are cooling a bunch with emittance $\epsilon_N = n\epsilon_{N,\min}$. Then, $\sigma^2_{\theta_x}$ is n times that given in eq. (3). To keep transport losses small, we will need to have angular acceptance $m\sigma_{\theta_x}$, where $m \gtrsim 4$. Together, we infer that the angular acceptance must be,

$$\theta_{x,\max} = m\sqrt{\frac{n(13.6 \text{ MeV})^2}{2\beta E\langle -dE_\mu/ds\rangle L_R}}. \tag{4}$$

  For example, with muons of $\gamma = 2$, $\beta = 0.866$, $E = 210$ MeV, and liquid hydrogen absorbers for which $\langle -dE_\mu/ds\rangle = 4.4$ MeV g^{-1}cm^2, and $L_R = 61.3$ g/cm^2, we have,

$$\theta_{x,\max} = m\sqrt{0.0019n} = 0.35 \text{ radian, for } n = m = 4. \tag{5}$$

  This angle is unusually large for particle beam transport, and emphasizes the technical challenge of ionization cooling. We also see that it will be almost impossible to design a single apparatus that cooled an initial emittance 10 times the minimum (1). Rather, the cooling channel should consist of a sequence of sections i, with focusing strength,

$$\beta_{\perp,i} = \frac{\beta_{\perp,0}}{i}, \tag{6}$$

  with lengths chosen so that section i cools from $4\epsilon_{N,i,\min}(\beta_{\perp,i})$ to $2\epsilon_{N,i,\min}(\beta_{\perp,i}) = 4\epsilon_{N,i+1,\min}(\beta_{\perp,i+1})$.

- **We should consider Robinson's "law" of damping decrements [12].**

  A version of this appeared in [4], and another appeared in [13], but it hasn't been discussed much since [14] in 1994.

  The idea is that if one pushes too hard on transverse cooling, one will inevitably be in a situation where longitudinal heating becomes very severe.

  Since I've never followed a derivation of this "law" before, I present one in sec. 3.3. See also [15].

- **We should consider frictional cooling.**

  However, I don't have the reference for this.

- **We should consider the problem of space-charge and wakefield effects.**

  In particular, has the work of Christine Celata appeared in referenceable form?

  Are there any referenceable note on wakefield effects?

- **Why do we use a solenoid channel to confine the muons during cooling (+ capture and phase rotation)?**

  This question seems to come up over and over as new people start thinking about a Muon Collider. So, although the answer is relatively straightforward, it would be good to include it. My version follows. See also [10, 17].

  Equation (1) shows that the absorbers should be placed at a low-$\beta_\perp$ point, *i.e.*, where the confining forces are as strong as possible. Between the absorbers we must have accelerating cavities, whose transverse dimensions are relatively large. To keep costs down, the confining forces should be weaker in the accelerating cavities. Hence, we need a confining structure with a periodic variation in the strength of the confining forces.

  We also wish to transport with low losses a bunch with extremely large transverse emittance, *i.e.*, one with particles at large values of the transverse coordinates. Use of a quadrupole channel to provide the confining forces would result in large numbers of particles passing through the quadrupole fringe fields at large radii, leading to significant interaction with the higher-order terms in the transport equations, and poor transport efficiency. In contrast, a solenoid channel provides confining forces that are essentially continuous in z and which are independent of radius. This permits effective transport over the full aperture.

  Of course we must still address the following:

- **Why do we use alternating solenoids?**

  Bob's recent note [16] is an important step toward understanding this question, and should be referenced in the Status Report (after some technical oversights are fixed).

  Indeed, on reading that note, I come to several conjectures beyond Bob's conclusions. These follow from two basic observations:

  - If we desire particles to have zero mechanical angular momentum after exiting a solenoid, they must have zero canonical angular momentum when inside the solenoid. (These angular momenta are measured with respect to the symmetry axis of the magnetic field.)
  - For a particle to have zero canonical angular momentum when inside a solenoid, the trajectory must pass through the symmetry axis of the magnet.

  The first observation follow from conservation of canonical (rather than mechanical) angular momentum; the second is discussed in sec. 4 along with other related issues.

  We draw several conclusions:

  1. Note that if the primary target were entirely along the magnetic symmetry axis, then all secondary particles would be produced with very low canonical angular momentum. That is, we could arrange for the beam to start with the desired special condition. Our task would then be to keep from generating canonical angular momentum later.
  2. Some canonical angular momentum is generated when the pion decays. (If the pion decays at a point not on the magnetic symmetry axis, the neutrino carries some angular momentum.)
  3. If the muon passes through an absorber when its trajectory is away from the magnetic axis, is canonical angular momentum is changed. There is some probability that the canonical angular momentum is "heated" even though the mechanical transverse momentum is "cooled". Hence, it is possible that more optimal cooling of angular momentum would involve absorbers whose thickness decreases with radius.
  4. A speculation that might be worthy of simulation is that a cooling channel with a single sign of the magnetic field could be designed that emphasizes keeping the canonical angular momentum small.
  5. If this is only partially successful, it might still be the case that a single reversal of the magnetic field near the end of the channel would suffice.

- **Shouldn't we give more details as to the sophistication of the simulations?**

  For example, mention the recent efforts to understand tails of the multiple scattering distribution. The present text could leave the impression we are still working in the Gaussian approximation.

- **What is the relation between the rf frequency and the emittance?**

  We have established that higher magnetic fields are needed to cool lower transverse emittances. As I understand it, the rf frequency is related to the bunch length, which is related to the longitudinal emittance.

  Roughly, I expect that the rf frequency will increase as we move down the cooling channel. Do we need to up the rf frequency every time we up the magnetic field?

I note that the two examples, VD and VE, use the same rf frequency while the field of E is twice that of D, but no comment is made here about the broader issue.

In the first paragraph of sec. VG, I read "It should be pointed out that in the earlier and later stages...." This sentence is very interesting. It not only should be pointed out, it should be explained! Why does the bunch length grow in the "later stages"? Later than what? The 31-T example is presented as the last stage of a Higgs factory. Perhaps the bunch lengthening in "later" stages is a holdover from earlier designs in which we skipped longitudinal emittance exchange near the end. I was never clear as to how well motivated that choice was. Is it still active? Discussion is warranted!

- **Are wedges the best way to deal with the problem of longitudinal/transverse emittance exchange?**

  We now have separate simulations of 6-d cooling by a factor of 2 in 20 meters of an alternating solenoid channel, and of exchange of longitudinal and transverse phase space in a double bent solenoid section. Each of these requires rather particular phase-space correlations for efficient operation.

  As I understand it, a procedure for "matching" from one set of correlations to another does not yet exist, or requires a very long distance (hence, significant cost in dollars and in muon decay loss).

  Further, as near as I can tell, any system involving wedges and momentum dispersion in dipoles or bent solenoids requires separate cooling channels for positive and negative muons. This doubles the cost of the cooling channel compared to that needed for one sign. Since my estimate of the cost for a single cooling channel is a large fraction of $1B, this is a major effect.

  Hence, **finding a new solution to the emittance-exchange problem is the major issue in muon collider design in the near future.**

  But, the Status Report gives little indication that we are aware of this issue.

  My conclusion is that some of the late studies of the FOFO scheme deserve further investigation. I elaborate.

  It has long been realized [3, 24] that longitudinal phase space could be damped by passing higher-momentum particle through longer absorbers: *i.e.*, use wedge absorbers plus momentum dispersion.

  We also realize that transverse cooling only works well if there is a reasonably strong correlation between momentum and amplitude. That is, transverse cooling requires a kind of momentum dispersion.

  Hence, there appears to be a golden opportunity to use a requirement of transverse cooling to solve a major problem of longitudinal cooling. Namely, use absorbers whose thickness varies with radius to combine transverse cooling with continual longitudinal-transverse emittance exchange.

  This scheme has the major merit of working for both charges in a single cooling channel. This is a billion dollar savings, if it can be realized.

The hoped-for solution to the matching problem is not to match between two very different sets of correlations, but to find a single set of correlations that permits effectively simultaneously transverse and longitudinal cooling (the latter via emittance exchange rather than direct cooling).

Some time ago, Bob made a heroic effort to implement this notion. It seemed to me that success was close, when a problem of longitudinal instabilities emerged. As I understand it (there is no surviving documentation from this effort that I could find), the infamous synchrobetatron coupling was too strong in the FOFO scheme – and the whole idea was dropped.

It is far from obvious to me that the synchrobetatron problem had anything to do with the use of variable thickness absorbers.

However, the use of variable thickness absorbers has its price. The transverse cooling per meter will be less. But, the law of damping decrements advises us that any scheme that, in effect, cools simultaneously in all 3 subspaces will cool more slowly in some of those subspaces than a scheme that cools in some subspaces while heating in others.

It may well take twice as long to reduce the 6-d emittance by a factor of 2 with variable thickness absorbers, compared to the present scheme. But, no bent solenoid sections would be required, so the total length of the cooling channel would be about the same. More rf acceleration will be required.

[Recall that rather general arguments based on angular acceptance and cooling efficiency indicate that cooling apparatus of a given peak magnetic field and corresponding rf frequency should be used for cooling only over a factor of 2 in 2-d transverse emittance. Hence, a cooling "section" would be longer in the new scheme.]

The use of variable thickness absorbers might, however, aggravate the angular momentum problem. There is a hint (sec. 4) that the angular momentum problem would be better addressed if the absorbers were limited to small radii. This is a use of variable thickness absorbers, but not in the manner needed for emittance exchange. For the latter, the absorber must be thicker at larger radii, where the momenta are higher.

This may reinforce the need for the alternating solenoid scheme – which possibly is overly powerful for the simple case of uniform absorbers.

Recall that Bob's attempt to implement variable thickness absorbers took advantage of the variation of the betatron function along the FOFO lattice, and (I think) absorbers that were thickest on axis were placed that the high-$\beta_\perp$ points, while "complementary" absorbers that were thickest well off axis were placed at the low-$\beta_\perp$ points. The latter required absorbers inside the rf cavities. In that scheme, the absorbers were LiH, so they were relatively compact. If we use liquid hydrogen absorbers, the rf cavities would probably have to be split into two sections on either side of the low-$\beta_\perp$ points.

While my particular conjecture as to how cooling studies might evolve is perhaps too speculative for the Status Report, we should give more indication that we understand the need for improved cooling scenarios.

3 Derivation of the Cooling Equations

3.1 Transverse Cooling

I now attempt to derive SR(24).

The rms normalized 2-d transverse emittance in coordinates x and P_x of a bunch of particles moving along the z axis is related by,

$$m^2c^2\epsilon_N^2 = \left\langle x^2 \right\rangle \left\langle P_x^2 \right\rangle - \langle xP_x \rangle^2 . \tag{7}$$

Consider the propagation of the bunch in the z direction through a thin absorber. In the impulse approximation, the particles' momenta change, but their transverse positions do not. Hence, the rate of change of the ϵ_N^2 along z may be written,

$$2m^2c^2\epsilon_N \frac{d\epsilon_N}{dz} = \left\langle x^2 \right\rangle \left\langle \frac{dP_x^2}{dz} \right\rangle - 2\langle xP_x \rangle \left\langle x\frac{dP_x}{dz} \right\rangle . \tag{8}$$

3.1.1 $\langle xP_x \rangle = 0$

If we ignore correlations such as $\langle xP_x \rangle$, we obtain the simpler form,

$$2m^2c^2\epsilon_N \frac{d\epsilon_N}{dz} \approx \left\langle x^2 \right\rangle \left\langle \frac{dP_x^2}{dz} \right\rangle . \tag{9}$$

We wish to relate this to energy loss and multiple scattering caused by the absorber. We introduce the particle's angle to the z axis in the x-z plane,

$$\theta_x \approx \frac{P_x}{P_z} \approx \frac{P_x}{P}, \tag{10}$$

where the approximations suppose that P_x and P_y are much less than P_z. Then,

$$P_x \approx \theta_x P. \tag{11}$$

We also note that

$$cP = \beta E, \qquad \text{and} \qquad c^2P^2 = E^2 - (m_\mu c^2)^2, \tag{12}$$

leading to,

$$dP^2 = \frac{dE^2}{c^2} \qquad \text{and} \qquad \frac{dP}{dE} = \frac{E}{c^2P} = \frac{1}{v}. \tag{13}$$

Thus,

$$\begin{aligned} \frac{dP_x^2}{dz} &\approx \theta_x^2\frac{dP^2}{dz} + P^2\frac{d\theta_x^2}{dz} = \frac{\theta_x^2}{c^2}\frac{dE^2}{dz} + P^2\frac{d\theta_x^2}{dz} = 2\frac{\theta_x^2 E}{c^2}\frac{dE}{dz} + P^2\frac{d\theta_x^2}{dz} \\ &= 2\frac{P_x^2}{\beta^2 E}\frac{dE}{dz} + P^2\frac{d\theta_x^2}{dz}. \end{aligned} \tag{14}$$

We average eq. (14) over the bunch, but suppose that we may replace β, E P by their bunch averages. In effect, we are neglecting correlations between transverse momentum and total momentum. We have already noted in sec. 2 that the existence of such correlations is, however, critical to the success of cooling. But for correlations that are "not too large", we proceed,

$$\left\langle \frac{dP_x^2}{dz} \right\rangle \approx 2\frac{\langle P_x^2\rangle}{\beta^2 E}\frac{dE}{dz} + \langle P\rangle^2 \left\langle \frac{d\theta_x^2}{dz} \right\rangle \approx 2\frac{\langle P_x^2\rangle}{\beta^2 E}\frac{dE}{dz} + \frac{(13.6\ \mathrm{MeV}/c)^2}{\beta^2 L_R}, \tag{15}$$

using the standard form for the rms projected multiple scattering (for example, (23.9) of [19]), where L_R is the radiation length of the absorber material.

When we insert eq. (15) into eq. (9), we find,

$$2m^2c^2\epsilon_N \frac{d\epsilon_N}{dz} \approx 2\frac{\langle x^2\rangle \langle P_x^2\rangle}{\beta^2 E}\frac{dE}{dz} + \frac{\langle x^2\rangle (13.6\ \mathrm{MeV}/c)^2}{\beta^2 L_R}. \tag{16}$$

In the first term on the right of eq. (16), the averages contain ϵ_N^2 according to eq. (7) with the neglect of correlations. In the second term, we follow Neuffer and write,

$$\left\langle x^2 \right\rangle = \epsilon\beta_\perp = \frac{\epsilon_N \beta_\perp}{\gamma\beta} \tag{17}$$

where ϵ is the rms 2-d geometric transverse emittance, and $\beta_\perp$ is the value of the betatron function of the transversely confining beam optics at the position of the absorber. Hence,

$$2m^2c^2\epsilon_N \frac{d\epsilon_N}{dz} \approx 2\frac{(mc)^2\epsilon_N^2}{\beta^2 E}\frac{dE}{dz} + \frac{\epsilon_N\beta_\perp (13.6\ \mathrm{MeV}/c)^2}{\beta^3\gamma L_R}, \tag{18}$$

and so,

$$\frac{d\epsilon_N}{dz} \approx \frac{\epsilon_N}{\beta^2 E}\frac{dE}{dz} + \frac{\beta_\perp (13.6\ \mathrm{MeV}/c)^2}{2\beta^3\gamma(m_\mu c)^2 L_R} = \frac{\epsilon_N}{\beta^2 E}\frac{dE}{dz} + \frac{\beta_\perp (13.6\ \mathrm{MeV}/c)^2}{2\beta^3 E m_\mu L_R}. \tag{19}$$

This confirms SR(24) of the Status Report, with sufficient neglect of correlations.

A picky point: in my derivation (and in the notation of the Particle Data Group [19]), dE/dz is negative. But in the notation of the Status Report, dE/ds is positive and is called the energy loss. The notation of the Status Report, while nonstandard in the larger community, was no doubt chosen so that the "cooling" term is easily identified by the minus sign in front of it.

It is instructive to use approximate analytic expressions for dE/dz and L_R so that the two terms of eq. (19) are more readily compared. The Bethe formula [18] for dE/dz is given as eq. (23.1) of [19],

$$\frac{dE}{dz} \approx -4\pi r_e^2 m_e c^2 N_0 \rho \frac{Z}{A}\left(\frac{1}{\beta^2}\ln\frac{2\gamma^2\beta^2 m_e c^2}{I} - 1\right). \tag{20}$$

Here, N_0 is Avagadro's number (per mole), ρ is the density of the absorber in, say, g/cm^3, the atomic "weight" A is in g/mole, and I is the ionization potential of the absorber material. We have set the maximum kinetic energy imparted to an electron in a collision with a muon to $2\gamma^2\beta^2 m_e c^2$, which is valid for $\gamma m_e/m_\mu \ll 1$, as holds in the muon cooling channel. We have also neglected the density-effect term, which is significant only for $\gamma \gtrsim 3$.

A useful fit for the radiation length L_R has been given as (23.19) of [19],

$$\frac{1}{L_R} = 4\alpha r_e^2 N_0 \rho \frac{Z(Z+1)}{A} \ln \frac{287}{\sqrt{Z}}, \tag{21}$$

where α is the fine-structure constant.

With eqs. (20-21), eq. (19) can also be written as,

$$\frac{d\epsilon_N}{dz} \approx \frac{1}{\beta^2 E L_R}\left[-\frac{\pi m_e c^2 \epsilon_N}{\alpha(Z+1)\ln\frac{287}{\sqrt{Z}}}\left(\frac{1}{\beta^2}\ln\frac{2\gamma^2\beta^2 m_e c^2}{I} - 1\right) + \frac{\beta_\perp (13.6\text{ MeV}/c)^2}{2\beta m_\mu}\right] \tag{22}$$

For example, with a hydrogen absorber we take $I = 15$ eV, and find,

$$\frac{d\epsilon_N}{dz} \approx \frac{1}{\beta^2 E L_R}\left[-19.2\epsilon_N\left(\frac{12}{\beta^2} - 1\right) + \frac{0.88\beta_\perp}{\beta}\right], \tag{23}$$

where the value 12 holds for $\gamma\beta \approx 2$. The minimum value of ϵ_N that can be achieved with a hydrogen absorber at a location where the betatron function is $\beta_\perp$ is then,

$$\epsilon_{N,\min} = \frac{0.0038\beta\beta_\perp}{1 - \beta^2/12}. \tag{24}$$

Equation (23) can be rewritten in terms of $\epsilon_{N,\min}$ as,

$$\frac{1}{\epsilon_N}\frac{d\epsilon_N}{dz} \approx -\frac{230\text{ MeV}(1-\beta^2/12)}{\beta^4 E L_R}\left(1 - \frac{\epsilon_{N,\min}}{\epsilon_N}\right). \tag{25}$$

For example, with $\gamma = 2$, $\beta = 0.866$, $E = 210$ MeV, $P = 182$ MeV/c, then

$$\frac{1}{\epsilon_N}\frac{d\epsilon_N}{dz} \approx -\frac{1.8}{L_R}\left(1 - \frac{\epsilon_{N,\min}}{\epsilon_N}\right). \tag{26}$$

To cool ϵ_N from, say, $4\epsilon_{N,\min}$ to $2\epsilon_{N,\min}$ would require about $0.6L_R \approx 480$ cm of liquid hydrogen, using eq. (26) with $\langle 1 - \epsilon_{N,\min}/\epsilon_N\rangle \approx 2/3$. I believe ICOOL indicates that about 600 cm would be required.

3.1.2 $\langle xP_x\rangle \neq 0$

In this case, we also need the average $\langle x dP_x/dz\rangle$, still in the impulse approximation, eq. (8). From eq. (15), we can write,

$$\frac{dP_x}{dz} \approx \frac{P_x}{\beta^2 E}\frac{dE}{dz} + \frac{(13.6\text{ MeV}/c)^2}{2P_x\beta^2 L_R}. \tag{27}$$

The second term, however, is ill behaved as $P_x \to 0$. Ignoring this, we would then find,

$$\left\langle x\frac{dP_x}{dz}\right\rangle \approx \frac{\langle xP_x\rangle}{\beta^2 E}\frac{dE}{dz} + \left\langle\frac{x}{P_x}\right\rangle\frac{(13.6\text{ MeV}/c)^2}{2\beta^2 L_R}. \tag{28}$$

For this to make physical sense, we must declare that $\langle x/P_x \rangle = 0$. This also can be justified as follows. A nonzero correlation $\langle xP_x \rangle$ in the initial particle distribution means that $\langle P_x \rangle$ varies with x. After passing through the thin absorber, the P_x are smeared by multiple scattering, but at a given x, the $\langle P_x \rangle$ remains unchanged, and so the correlation $\langle xP_X \rangle$ is unchanged by multiple scattering.

Hence, eq. (16) is now,

$$
\begin{aligned}
2m^2c^2\epsilon_N \frac{d\epsilon_N}{dz} &\approx 2\frac{(\langle x^2 \rangle \langle P_x^2 \rangle - \langle xP_x \rangle^2)}{\beta^2 E}\frac{dE}{dz} + \frac{\langle x^2 \rangle (13.6 \text{ MeV}/c)^2}{\beta^2 L_R} \\
&= 2\frac{(mc)^2 \epsilon_N^2}{\beta^2 E}\frac{dE}{dz} + \frac{\langle x^2 \rangle (13.6 \text{ MeV}/c)^2}{\beta^2 L_R},
\end{aligned}
\tag{29}
$$

using eqs. (7-8). In the presence of a correlation $\langle xP_x \rangle$, it would not be proper to use eq. (17). So we would just write,

$$
\frac{d\epsilon_N}{dz} \approx \frac{\epsilon_N}{\beta^2 E}\frac{dE}{dz} + \frac{\langle x^2 \rangle (13.6 \text{ MeV}/c)^2}{2\epsilon_N \beta^3 E m L_R}. \tag{30}
$$

However, without the simple relation eq. (17), we do not obtain as much insight from this equation as we can from eq. (1), which holds when $\langle xP_x \rangle = 0$.

3.1.3 Thick Absorbers

Thus far we have assumed the absorber is thin, and made the impulse approximation that a particle's x (and y) are unchanged during passage through the absorber. The case of thick absorbers has been considered by Juan and Rick in [6], with the general conclusion that if the confining fields are "strong enough", there is little qualitative change in the form of the transverse cooling equation.

3.2 Longitudinal Cooling

The argument here is little different from that in [3].

In the thin absorber limit there is no change in z of a particle as it cross an absorber. So it suffices to consider changes in P_z, or nearly equivalently, in E. More precisely, since the central energy E_0 is nonzero, we consider changes $\Delta E = E - E_0$ and desire an expression for $\langle d(\Delta E)^2/dz \rangle$.

There are two effects to consider: the variation in the mean energy loss with particle energy, and fluctuations about the mean. We calculate these separately, and add them in quadrature. First, a particle of energy E that traverses an absorber of thickness δz has mean energy loss δE_{mean} given by,

$$
\delta E_{\text{mean}} = \frac{dE}{dz}\delta z \approx \left(\frac{dE_0}{dz} + \Delta E \frac{d^2 E_0}{dE dz} \right) \delta z. \tag{31}
$$

The change in ΔE is then,

$$
\delta(\Delta E)_{\text{mean}} \approx \Delta E \frac{d^2 E_0}{dE dz} \delta z. \tag{32}
$$

Hence,

$$\frac{d(\Delta E)^2_{\rm mean}}{dz} \approx 2(\Delta E)^2 \frac{d^2 E_0}{dE dz}, \tag{33}$$

Second, we consider fluctuations in the energy loss in the absorber. To the first approximation, it suffices to consider this only for the central energy E_0. Using the nomenclature "straggling" for this effect, we have an additional term.

$$\frac{d(\Delta E)^2_{\rm straggling}}{dz}. \tag{34}$$

Combining this with eq. (33), we have,

$$\frac{d(\Delta E)^2}{dz} \approx 2(\Delta E)^2 \frac{d}{dE}\frac{dE_0}{dz} + \frac{d(\Delta E)^2_{\rm straggling}}{dz}. \tag{35}$$

This is eq. SR(25) of the Status Report, again noting that my dE/dz has the opposite sign to that used there.

3.2.1 $d(\Delta E)^2_{\rm straggling}/dz$

It is hard to find a crisp reference for the form of the energy straggling fluctuations. The basic calculation is due to Bohr [20]. Our past reference has been to Fano [21], but this paper is quite hard to read. It might be better to point to sec. 13.3 of [22], or at least to add this reference. Neuffer [14] quotes the desired result as,

$$\frac{d(\Delta E)^2_{\rm straggling}}{dz} = 4\pi (r_e m_e c^2)^2 N_0 \frac{Z}{A}\rho\gamma^2(1-\beta^2/2) = 2\pi (r_e m_e c^2)^2 N_0 \frac{Z}{A}\rho(\gamma^2+1), \tag{36}$$

to display the dependence on muon energy. This result applies only for "thick" absorbers, which is reasonable for the muon collider where we take the energy away 30 times by ionization loss, although each absorber is only about 5% of a radiation length.

When the muons are later accelerated, ΔE remains constant. Thus, we seek to minimize ΔE and not $\Delta E/E$. Hence, the undesirable "heating" due to straggling is minimized by operating at the lowest possible γ, according to eq. (36).

For the record, I note that eq. (36) can be recast in a way that emphasizes the radiation length L_R of the absorber by using the fit (23.19) of [19] (given above as eq. (21)),

$$\frac{d(\Delta E)^2_{\rm straggling}}{dz} = \frac{\pi (m_e c^2)^2 A(\gamma^2+1)}{2\alpha(Z+1)L_R \ln(287/\sqrt{Z})}. \tag{37}$$

Equation SR(25) of the Status Report could thus be written,

$$\frac{d(\Delta E)^2}{dz} \approx 2(\Delta E)^2 \frac{d}{dE}\frac{dE_0}{dz} + \frac{\pi (m_e c^2)^2 (\gamma^2+1)}{2\alpha(Z+1)L_R \ln(287/\sqrt{Z})}. \tag{38}$$

A sense of the relative importance of the two terms in eq. SR(25) can be gotten from the Bethe formula (20). With $E = \gamma m_\mu c^2$, the leading term in the derivative of eq. (20) with respect to E is,

$$\frac{d}{dE}\frac{dE}{dz} \approx 8\pi r_e^2 \frac{m_e c^2}{mc^2} N_0 \frac{Z}{A}\frac{\rho}{\gamma^3\beta^4}\left[\ln\frac{2\gamma^2\beta^2 m_e c^2}{I} - \gamma^2\right] \approx 8\pi r_e^2 \frac{m_e c^2}{m_\mu c^2} N_0 \frac{Z}{A}\frac{\rho}{\gamma^3\beta^4}(12-\gamma^2), \tag{39}$$

where the final approximation assumes $I \approx 15$ eV for the ionization potential of hydrogen. (This puts the dE/dz minimum at $\gamma = \sqrt{12}$, which is a bit low.)

Then, recalling eq. (36), eq. (35) becomes,

$$\begin{aligned}\frac{d(\Delta E)^2}{dz} &\approx 2\pi (r_e m_e c^2)^2 N_0 \frac{Z}{A}\rho \left[\frac{(\Delta E)^2}{m_e c^2 m_\mu c^2}\frac{4(12-\gamma^2)}{\gamma^3\beta^4} + (\gamma^2+1)\right] \\ &\approx \frac{\pi (m_e c^2)^2}{2\alpha(Z+1)\ln(287/\sqrt{Z})L_R}\left[\frac{48(\Delta E)^2}{m_e c^2}\frac{(1-\gamma^2/12)}{\gamma^2\beta^4 E} + (\gamma^2+1)\right]. \qquad (40)\end{aligned}$$

This form reveals that the heating due to the variation in dE/dz with energy is proportional to $1/\beta^4$, which is perhaps the strongest argument against cooling at very low β.

One or the other versions of eq. (40) would be a useful addition to the Status Report.

For a hydrogen absorber, we can write eq. (40) as,

$$\frac{1}{(\Delta E)^2}\frac{d(\Delta E)^2}{dz} \approx \frac{466\ \text{MeV}(1-\gamma^2/12)}{\gamma^2\beta^4 E L_R}\left(1 + \frac{1.1\ (\text{MeV})^2\gamma^3\beta^4(\gamma^2+1)}{(1-\gamma^2/12)(\Delta E)^2}\right). \qquad (41)$$

For example, with $\gamma = 2$ (which is about the largest we can consider for transverse cooling) and $\Delta E \approx 10$ MeV, the first term in eq. (41) is about twice the second, and,

$$\frac{1}{(\Delta E)^2}\frac{d(\Delta E)^2}{dz} \approx \frac{1}{L_R}. \qquad (42)$$

Recalling the example at the end of sec. 3.1.1, the transverse emittance ϵ_N was estimated to cool by a factor of 2 in $0.6L_R$. Equation (42) estimates that $(\Delta E)^2$ would grow by a factor of 1.8 over the same distance.

It looks to me like there is no value of γ for which the approximation eq. (40) predicts longitudinal ionization cooling. However, our approximation underestimates the slope of dE/dz for $\gamma > 3$, due to our neglect of the density effect.

It is noteworthy that cooling (heating) scales as the radiation length with coefficients near unity (for example, eqs. (26) and (42)). Perhaps we could say loosely that ionization cooling is a manifestation of the very low energy tail of bremsstrahlung, and is in some sense a form of radiative cooling. This suggests we can find other aspects of ionization cooling in common with radiative cooling, as in the following section.

3.3 The Law of Damping Decrements

I read in sec. 8.2.3, p. 287 of [23] that Robinson [12] showed that for a process that damps the 6-d emittance of a bunch, the sum of the damping decrements of all 3 2-d subemittances is a constant. *Robinson's paper does not give a general "proof", but an argument specific to radiative damping. Has someone else given a more general argument?*

In sec. 3.3.1, I look at momentum damping times (but don't complete the argument), and in sec. 3.3.2, I consider emittance damping times.

3.3.1 Momentum Damping Times

We follow [4] here.

If we ignore the "heating" effects of multiple scattering and straggling, an interesting relation between transverse and longitudinal "cooling" can be demonstrated.

In the first approximation, the effect of passage of a charged particle through an absorber is to reduce the magnitude of a particle's momentum $\mathbf{P}$ without changing its direction. That is,

$$\frac{d\mathbf{P}}{dt} = \frac{dP}{dt}\hat{\mathbf{P}}. \tag{43}$$

We note that,

$$\frac{dP}{dt} = \frac{dP}{dE}\frac{dE}{dz}\frac{dz}{dt} = \frac{v_z}{v}\frac{dE}{dz} \approx \frac{dE}{dz}, \tag{44}$$

recalling eq. (13).

The transverse part of eq. (43) can now be written,

$$\frac{dP_\perp}{dt} = \frac{dE}{dz}\frac{P_\perp}{P}, \tag{45}$$

so the transverse momentum is damped in time according to $\exp(-t/\tau_\perp)$, where,

$$\frac{1}{\tau_\perp} = -\frac{1}{P}\frac{dE}{dz}. \tag{46}$$

Recall that in our notation, $dE/dz < 0$. Remember also that muon cooling does not proceed by damping the total momentum P to zero. Rather, the energy lost to ionization is continually replenished by accelerating cavities between the absorbers, such that P remains essentially constant at some value P_0.

Turning to the longitudinal momentum, we are not concerned with damping P_z to zero, but damping the difference, $\Delta P_z = P_z - P_{z,0}$, while $P_{z,0} \approx P_0$. Much as in sec. 3.2, we then write,

$$\frac{d\Delta P_z}{dt} = \frac{d}{dP_z}\frac{dE}{dz}\Delta P_z \approx \frac{d}{dP}\frac{dE}{dz}\Delta P_z = \frac{dE}{dP}\frac{d}{dE}\frac{dE}{dz}\Delta P_z = v\frac{d}{dE}\frac{dE}{dz}\Delta P_z. \tag{47}$$

Thus, the longitudinal damping time $\tau_\parallel$ is given by,

$$\frac{1}{\tau_\parallel} = -\frac{d}{dP}\frac{dE}{dz} = -v\frac{d}{dE}\frac{dE}{dz}. \tag{48}$$

For $\gamma < 3$-4, $\tau_\parallel < 0$, the longitudinal momentum spread is not damped, but grows with time.

The above is true, but are we making any money from it?

3.4 Emittance Damping Times

This section follows [13].

We could also consider the damping of the emittances, again with the neglect of multiple scattering and straggling.

Thus, eq. (19) tells us that the transverse emittance ϵ_N has a damping distance $z_\perp$ given by,

$$\frac{1}{z_\perp} = -\frac{1}{\beta^2 E}\frac{dE}{dz}. \tag{49}$$

When a particle traverse one damping distance in the absorber, it loses energy $E_\perp$ related by,

$$\frac{1}{E_\perp} = -\frac{1}{z_\perp \frac{dE}{dz}} = \frac{1}{\beta^2 E}. \tag{50}$$

Similarly, eq. (33) for $(\Delta E)^2$ leads to a damping distance $z_{\Delta E}$ given by,

$$\frac{1}{z_{\Delta E}} = -2\frac{d}{dE}\frac{dE}{dz}, \tag{51}$$

and the energy loss $E_{\Delta E}$ over this distance is,

$$\frac{1}{E_{\Delta E}} = \frac{2\frac{d}{dE}\frac{dE}{dz}}{\frac{dE}{dz}} \approx -\frac{2}{\gamma^2\beta^2 E}\left(1 - \frac{\gamma^2}{12}\right), \tag{52}$$

where the approximation follows from eq. (20-39).

We now consider the sum of the energy damping decrements of the 2-d emittances in x, y and ΔE. Noting that $E_x = E_y = E_\perp$, we have,

$$\sum \frac{1}{E_i} = \frac{2}{E_\perp} + \frac{1}{E_{\Delta E}} = \frac{2}{\beta^2 E} + \frac{2\frac{d}{dE}\frac{dE}{dz}}{\frac{dE}{dz}} \approx \frac{2}{E}\left(1 + \frac{1}{12\beta^2}\right) \approx \frac{2}{E}, \tag{53}$$

where the first approximation is based on eq. (52), and the second approximation is reasonable for β near 1.

It is implied in [13] that the final result of eq. (53) is exact, but my derivation is not powerful enough to reveal this.

In our case, the claim of Robinson could be satisfied by any function of E, not just $2/E$. However, it is helpful to know that the result eq. (53) is not an accident.

The impact of eq. (53) for muon cooling is that strong transverse cooling implies strong longitudinal heating, even when neglecting multiple scattering and straggling!

4 Canonical Angular Momentum

4.1 Kinematic Facts

The canonical angular momentum of a charge e, with mechanical momentum $\mathbf{P} = (P_r, P_\phi, P_z)$ in a solenoid field $\mathbf{B} = B_z\hat{\mathbf{z}}$ is,

$$L_z = r\left(P_\phi + \frac{eA_\phi}{c}\right) = rP_\phi + \frac{er^2 B_z}{2c}, \tag{54}$$

in Gaussian units, where r is the distance from the magnetic symmetry axis in cylindrical coordinates (r, ϕ, z), and the (coulomb-gauge) vector potential of the solenoid field is $\mathbf{A} =$

$(0, rB_z/2, 0)$. If the particle has transverse momentum $P_\perp$, the radius R_B of its helical trajectory in the magnetic field is,

$$R_B = \frac{cP_\perp}{eB_z}, \tag{55}$$

and the sense of rotation of the trajectory is $-\hat{\mathbf{z}}$ (Lenz's law).

We label R_G as the distance from the magnetic axis to the "guiding ray" of the helical trajectory (axis of the helix).

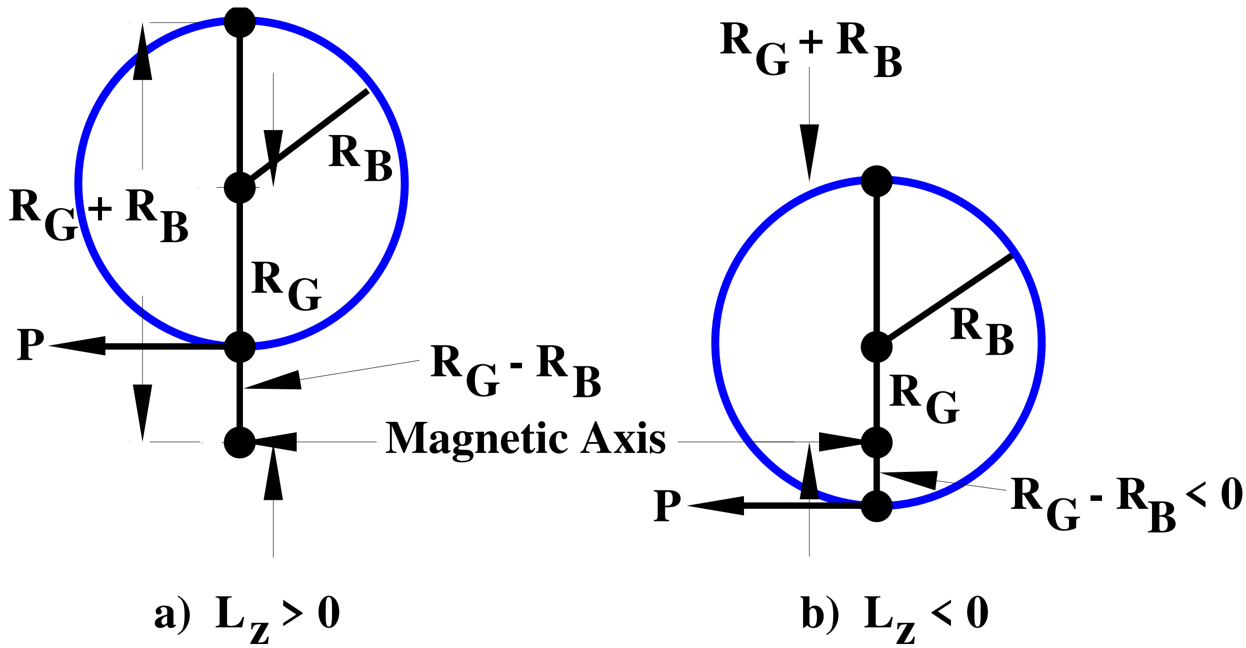

Figure 1: The projection onto a plane perpendicular to the magnetic axis of the helical trajectory a charge particle of transverse momentum P. The magnetic field B_z is out of the paper, so the rotation of the helix is clockwise for a positively charged particle. a) The trajectory does not contain the magnetic axis, and $L_z > 0$. b) The trajectory contains the magnetic axis, and $L_z < 0$.

Since the canonical angular momentum is a constant of the motion, we can evaluate it at any convenient point on the particle's trajectory. In particular, we consider the point at which the trajectory is closest to the magnetic axis. As shown in Fig. 1, this point obeys $r = R_G - R_B$, and so eq. (54) tells us that,

$$L_z = (R_G - R_B)\, P_\perp + \frac{eB_z}{2c} (R_G - R_B)^2 = \left(R_G^2 - R_B^2\right) \frac{eB_z}{2c}, \tag{56}$$

using eq. (55). Note that $R_G^2 - R_B^2$ is the product of the closest and farthest distances between the trajectory and the magnetic axis.

Hence, the canonical angular momentum vanishes for motion in a solenoid field if and only if $R_G = R_B$, *i.e.*, if and only if the particle's trajectory passes through the magnetic axis.

We also see that if the trajectory does not contain the magnetic axis, the canonical angular momentum is positive; while if the trajectory contains the magnetic axis, the canonical angular momentum is negative.

4.2 The Effect of Energy Loss in an Absorber

We "cool" the transverse momentum of the muon with absorbers placed in their path. Then, the projected trajectory of the muon after the absorber is a circle of smaller radius, according to eq. (55). In the first approximation, we ignore multiple scattering. Then, the circle after the absorber is tangent to the circle before the absorber, and the point of tangency is at the absorber, as shown in Fig. 2.

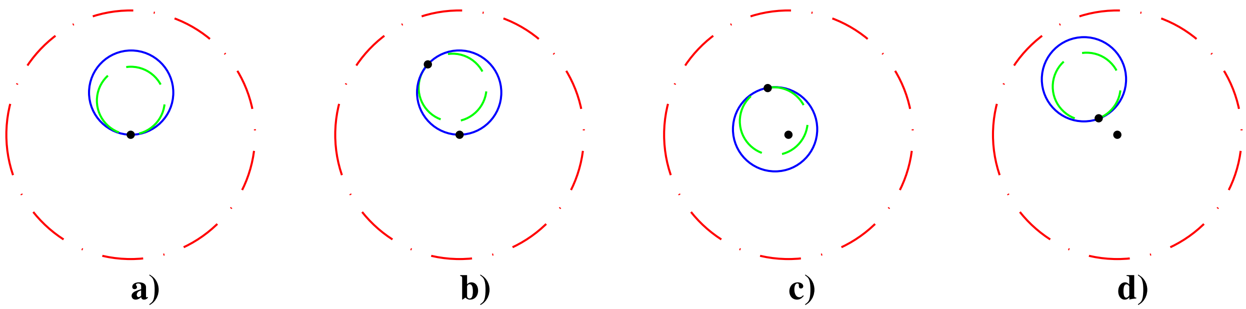

Figure 2: The effect of energy loss in an absorber on particle trajectories in a solenoid magnetic field. The outer (dash-dot) circles represent the coil of the magnet, seen end-on. The solid and dashed circles are the projections of particle's trajectory before and after passing through the absorber, respectively. a) The initial trajectory has zero canonical angular momentum, and therefore passes through the magnetic axis. The point of absorption is on the axis also. The final transverse momentum is lower, but the canonical angular momentum is still zero. b) The canonical angular momentum is initially zero, but becomes nonzero after the absorber. c) The canonical angular momentum is initially nonzero, but the magnetic axis is within the trajectory. The absorber reduces the magnitude of the canonical angular momentum. d) The canonical angular momentum is initially nonzero, and the magnetic axis is not within the trajectory. The absorber decreases or increases the canonical angular momentum, depending on the location of the absorber.

There are several cases:

1. The initial canonical angular momentum is zero, and the absorber is on the magnetic axis (Fig. 1a). The canonical angular momentum remains zero.

2. The initial canonical angular momentum is zero, but the absorber is not on the magnetic axis (Fig. 1b). The final trajectory does not contain the magnetic axis, and the final canonical angular momentum is greater than zero.

3. The initial canonical angular momentum is less than zero, so the trajectory contains the magnetic axis (Fig. 1d). The final trajectory lies within the initial trajectory. Unless the energy loss is large, the final trajectory will still contain the magnetic axis, and the final canonical angular momentum remain negative. The magnitude of the closest and farthest distances of the trajectory from the axis are both reduced, so the magnitude of the canonical angular momentum is reduced.

4. The initial canonical angular momentum is greater than zero, so the trajectory does not contain the magnetic axis (Fig. 1d). The final trajectory lies within the initial trajectory, so the canonical angular momentum remains positive and its value can either increase or decrease. If the absorber is close to the magnetic axis, the closest distance of the trajectory to the magnetic axis is little changed, but the farthest distance is decreased; hence, the canonical angular momentum would decrease. However, if the absorber is far from to the magnetic axis, the farthest distance of the trajectory to the magnetic axis is little changed, while the closest distance is increased hence, the canonical angular momentum would increase.

In general, the effect of a series of absorbers not on the magnetic axis would be to "cool" the transverse momentum $P_\perp$ and the trajectory radius R_B towards zero, but to leave the particle with a nonzero guide radius R_G, and hence with a nonzero canonical angular momentum.

One option is to restrict the absorbers to a maximum radius R_A. In the limit that R_B is cooled to zero, we would still have $R_G \approx R_A$. Hence the canonical angular momentum would converge on a value of order $eR_A^2 B_z/2c$. The particle would emerge from the solenoid with this value for its mechanical angular momentum. Hence, eq. (56) tells us,

$$P_{\phi,\mathrm{outside}}[\mathrm{MeV}/c] \approx \frac{eR_A B_z}{2c} = 150 R_A[\mathrm{m}] B_z[\mathrm{T}]. \tag{57}$$

For example, if $R_A = 1$ cm and $B_z = 15$ T, then $P_{\phi,\mathrm{outside}} \approx 22.5$ MeV/c. This is still rather high.

Another option is to use alternating solenoids.

4.3 Alternating Solenoids

This section follows [16]. *Bob's definition of canonical angular momentum has a sign error for the term involving the magnetic field, if charge e is taken as positive.*

The possible advantage of alternating the direction of the magnetic field in the solenoids is sketched in Fig. 3.

Suppose the transverse momentum of a particle has been cooled to zero, but the radius of the guiding ray is nonzero, say $R_G = R_0$. The radius R_B of the helical trajectory is zero. The canonical angular momentum of that particle in a field $B_z = +B$ is $L_z = +eR_0^2 B/2c$. The projection of the motion on a plane perpendicular to $\mathbf{B}$ is shown in Fig. 3a.

If the particle then exited the solenoid, the situation would be as sketched in Fig. 3b. In the impulse approximation, the radius of the particle does not change, but the fringe field of the solenoid gives it an azimuthal kick. This is easily calculated via conservation of canonical angular momentum, and we find that $P_{\phi,\mathrm{outside}} = eR_0 B/2c$. This is very undesirable.

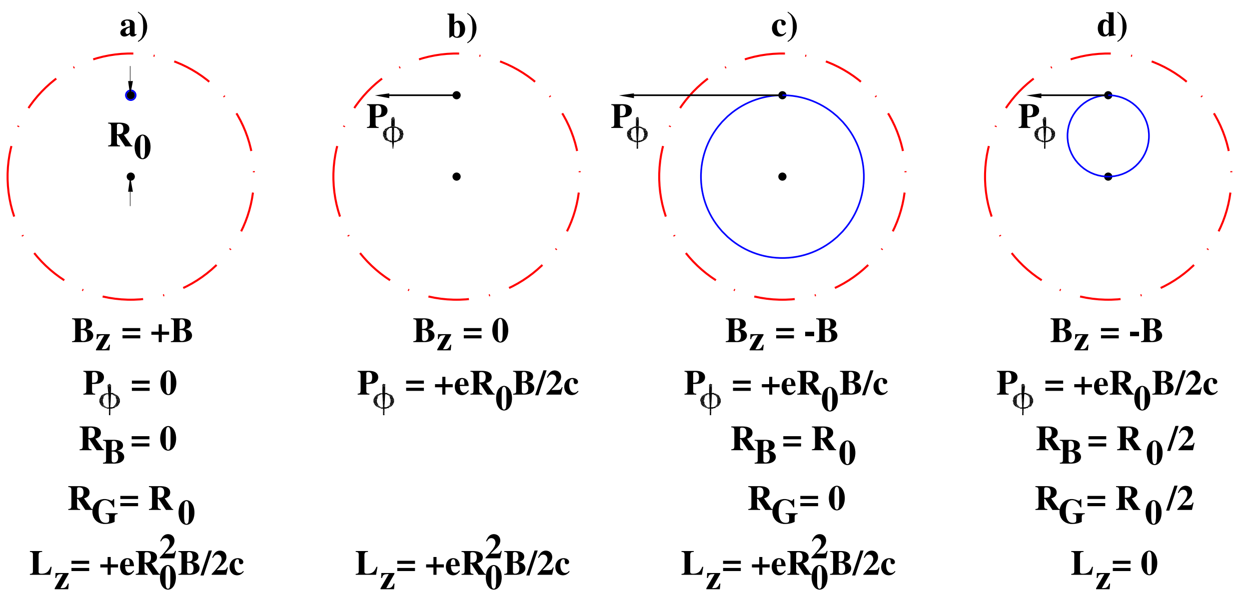

Figure 3: The effect of alternating the sign of the magnetic field in the solenoid.

Suppose instead, the particle exited the solenoid with $B_z = +B$ and immediately entered another solenoid with $B_z = -B$, as sketched in Fig. 3c. Again the canonical angular momentum of the particle is conserved, and in the impulse approximation, the position of the particle does not change. Hence, the transverse momentum is kicked by the fringe fields to double the amount in case b), namely $P_\phi = eR_0^2B/c$. Since the particle is in a magnetic field, its projected trajectory is a circle, but now $R_B = R_0$, and the circle is centered on the magnetic axis, so that $R_G = 0$.

If we can cool the newly created transverse momentum to half its value while in the field $B_z = -B$, then the helix radius will shrink to $R_B = R_0/2$. If in this process, the radius of the guiding ray rises from zero to $R_0/2$, then the final canonical angular momentum would be zero and the particle could exit the magnet without experiencing an azimuthal kick. This is shown in Fig. 3d.

If we have enough control over the growth of R_G during the cooling to guarantee the desired final condition, $R_G = R_0/2$, a single reversal of the solenoid field would suffice. This scenario has not been explored yet in simulation.

Rather, the present thinking is that frequent reversal of the solenoid field will best accomplish the desired goal of ending with very small canonical angular momentum. An ICOOL(?) simulation, Fig. 4, shows hows a ionization cooling in a sequence of alternating solenoids can keep the canonical angular momentum always near zero, while the mechanical angular momentum drops by a factor of two in 20 m.

5 Betatron Function of a Solenoid

Here I try to reconcile the language of betatron functions with a separate understanding of helical trajectories in a solenoid.

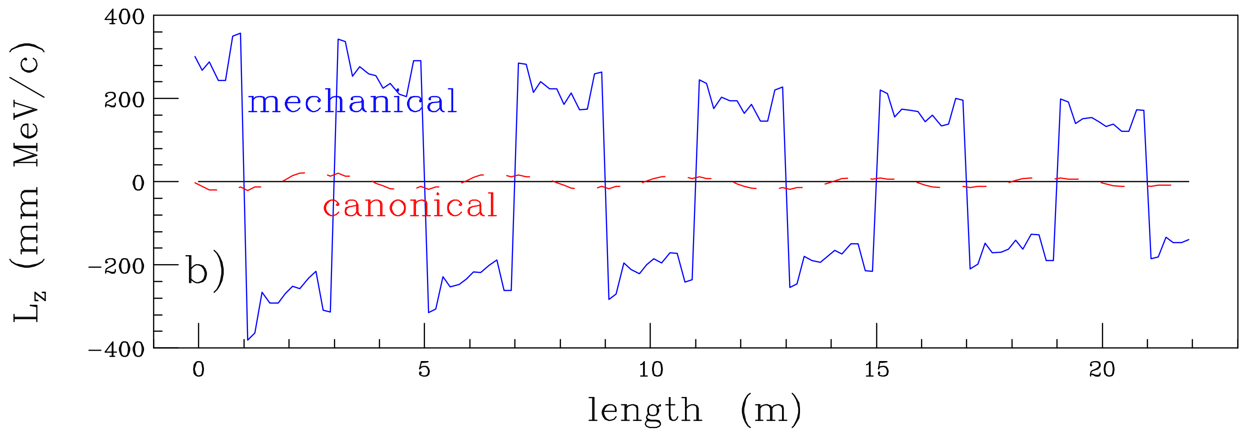

Figure 4: Cooling of angular momentum in a channel of solenoids whose fields are reversed every 2 m.

I read (in, for example, sec. 21 of [19]) that the amplitude $x(z)$ for a transverse coordinate of a particle in a beam transport system along the z axis is represented as,

$$x(z) = A\sqrt{\beta_\perp(z)}\cos(\phi(z) + \delta), \tag{58}$$

where A and δ are constants, $\beta_\perp$ is the betatron function, and ϕ is the phase-advance function which obeys,

$$\frac{d\phi}{dz} = \frac{1}{\beta_\perp}. \tag{59}$$

The projected slope $x'(z)$ of the trajectory then obeys,

$$\begin{aligned} x'(z) &= -\frac{A}{\sqrt{\beta_\perp(z)}}\sin(\phi(z) + \delta) + A\frac{\beta'_\perp(z)}{2\sqrt{\beta_\perp(z)}}\cos(\phi(z) + \delta), \\ &\approx -\frac{A}{\sqrt{\beta_\perp(z)}}\sin(\phi(z) + \delta), \end{aligned} \tag{60}$$

where the approximation holds for "slowly varying" betatron functions.

For the record, if we consider a bunch of particles with various A_i, then the rms bunch parameters are,

$$\sigma_x = \sigma_A\sqrt{\beta_\perp}, \qquad \sigma_{x'} = \frac{\sigma_A}{\sqrt{\beta_\perp}}, \qquad \text{and} \qquad \epsilon = \sigma_x\sigma_{x'} = \sigma_A^2, \tag{61}$$

ignoring correlations between x and x'. The usual relations follow,

$$\epsilon = \frac{\sigma_x}{\sigma_{x'}}, \qquad \sigma_x = \sqrt{\epsilon\beta_\perp}, \qquad \text{and} \qquad \sigma_{x'} = \sqrt{\frac{\epsilon}{\beta_\perp}}. \tag{62}$$

5.1 Uniform Solenoid

We first consider a solenoid with a uniform field B_z. The a particle with charge e and transverse momentum $P_\perp$ moves in a helix of radius,

$$R_B = \frac{cP_\perp}{eB_z}, \tag{63}$$

and with angular velocity (the Larmor frequency),

$$\omega_B = \frac{e\beta_\perp B_z}{P_\perp} = \frac{ecB_z}{E} = \frac{e\beta_z B_z}{P_z}. \tag{64}$$

I remember things like this from a 'relativistic' form of $F = ma$ for circular motion due to the Lorentz force: $\gamma m v_\perp^2/r = ev_\perp B_z/c$.

Of course, the particle moves in z with velocity $\beta_z c$. Hence, the x projection of a helix centered on the z is then,

$$x = R_B \cos(\omega_B t + \delta) = R_B \cos\left(\frac{\omega_B z}{\beta_z c} + \delta\right) = R_B \cos\left(\frac{eB_z z}{cP_z} + \delta\right), \tag{65}$$

Comparing with eqs. (58-59), it is natural to identify the betatron "function" as,

$$\beta_\perp = \frac{cP_z}{eB_z}, \tag{66}$$

based on the form of the phase (not of the amplitude). This result was quoted earlier as eq. (2).

We can, of course, write (63) as,

$$R_B = \frac{cP_z}{eB_z}\frac{P_\perp}{P_z} = \sqrt{\frac{cP_z}{eB_z}\frac{P_\perp}{P_z}}\sqrt{\beta_\perp}, \tag{67}$$

so the constant factor A (whose dimensions are [length]$^{1/2}$) in (58) is,

$$A = \sqrt{\frac{cP_z}{eB_z}\frac{P_\perp}{P_z}} = \sqrt{\frac{cP_\perp^2}{eB_z P_z}}. \tag{68}$$

5.2 Slowly Varying Field $B_z(z)$

For a solenoid whose field varies "slowly" in z, it is reasonable to define the local radius $R_B(z)$ of the (nonuniform) helix and the local Larmor frequency $\omega_B(z)$. Then, eq. (65) continues to have meaning, and again we identify the betatron function as eq. (66).

But does the radius of the helix obey the form $R_B(z) = A\sqrt{\beta_\perp(z)}$, as required for description eq. (58) to apply? Our use of the betatron function in eq. (2) emphasized this aspect.

For motion in a "slowly varying" field, the magnetic flux through the orbit is an adiabatic invariant

$$R_B^2 B_z = \frac{c^2 P_\perp^2}{e^2 B_z} = K = \text{ constant}, \tag{69}$$

using (63). Inserting eq. (69) in eq. (68), we have,

$$A = \sqrt{\frac{eK}{cP_z}} \approx \sqrt{\frac{eK}{cP}} = \text{ constant}, \tag{70}$$

since the total mechanical momentum P is constant in any static magnetic field.

We conclude that the description (58) of both amplitude and phase in terms of the betatron function (66) is valid for motion in a slowly varying solenoidal field so long as $P_z \approx P$, *i.e.*, so long as the angles θ_x and θ_y are not too large.

I am intrigued by a feature of the above discussion. We have shown that the form (58) is a reasonable description for the projected trajectory of a charged particle in a "slowly varying" solenoid field. Nowhere in the argument was there a requirement that the solenoid field be periodic in z. Yet the classic derivation of eq. (58) makes heavy use of this assumption. Section 5.5 of [25] gives a good discussion eq. of (58) as an insightful guess to the solution of the differential equation $x'' + k(z)x = 0$, without requiring $k(z)$ to be periodic.

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