Understanding Emittance Growth in Drifts

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Ring Cooler/Emittance Exchange Workshop

Basic Dynamics in Drift

- Governed by the Hamiltonian

$$-p_s = -\sqrt{(E/c)^2 - (mc)^2 - p_x^2 - p_y^2}$$

- Equations of motion

$$\frac{dx}{ds} = \frac{p_x}{\sqrt{(E/c)^2 - (mc)^2 - p_x^2 - p_y^2}}$$
$$\frac{dy}{ds} = \frac{p_y}{\sqrt{(E/c)^2 - (mc)^2 - p_x^2 - p_y^2}}$$
$$\frac{dt}{ds} = \frac{E}{c^2\sqrt{(E/c)^2 - (mc)^2 - p_x^2 - p_y^2}}$$

- Not linear in phase space variables

Basic Dynamics in Drift (cont.)

- Can integrate exactly:

$$x(s_1) = x(s_0) + \frac{p_x(s_1 - s_0)}{\sqrt{(E/c)^2 - (mc)^2 - p_x^2 - p_y^2}}$$

$$y(s_1) = y(s_0) + \frac{p_y(s_1 - s_0)}{\sqrt{(E/c)^2 - (mc)^2 - p_x^2 - p_y^2}}$$

$$t(s_1) = t(s_0) + \frac{E(s_1 - s_0)}{c^2\sqrt{(E/c)^2 - (mc)^2 - p_x^2 - p_y^2}}$$

Emittance Definition

- Defined in terms of second order moments

$$\Sigma(s) = \int \boldsymbol{z}\boldsymbol{z}^T \rho(\boldsymbol{z}, s) d^6\boldsymbol{z}$$

  - $\boldsymbol{z}$ is phase space variable vector
  - ρ is phase space density
- Under symplectic *linear* transforms $\boldsymbol{z}(s') = M(s', s)\boldsymbol{z}(s)$:

$$\Sigma(s') = \int \boldsymbol{z}\boldsymbol{z}^T \rho(\boldsymbol{z}, s') d^6\boldsymbol{z} = \int \boldsymbol{z}\boldsymbol{z}^T \rho(M^{-1}(s', s)\boldsymbol{z}, s) d^6\boldsymbol{z}$$

$$= M(s', s) \left[\int \boldsymbol{z}\boldsymbol{z}^T \rho(\boldsymbol{z}, s) d^6\boldsymbol{z} \right] M^T(s', s) = M(s', s)\boldsymbol{\Sigma}(s) M^T(s', s)$$

  - The second equality on the first line is Liouville's theorem
  - $\epsilon_6^2(s) = \det \Sigma(s)$; M has determinant 1; thus, ϵ_6 is preserved
  - Note that linear transforms transform ellipsoids into other ellipsoids

Emittance Definition: Individual Planes

- $M(s', s)$ can be written as $A(s')R(s', s)A(s)$, where R is 3 2x2 block rotations
  - Takes this form for ring or other periodic system
  - Transport line can often be written this way also
- Evolution of $\mathbf{\Sigma}$ under symplectic linear transform M becomes

$$A^{-1}(s')\mathbf{\Sigma}(s')JA(s')JR(s', s) = R(s', s)A^{-1}(s)\mathbf{\Sigma}(s)JA(s)J$$

- Hints at procedure

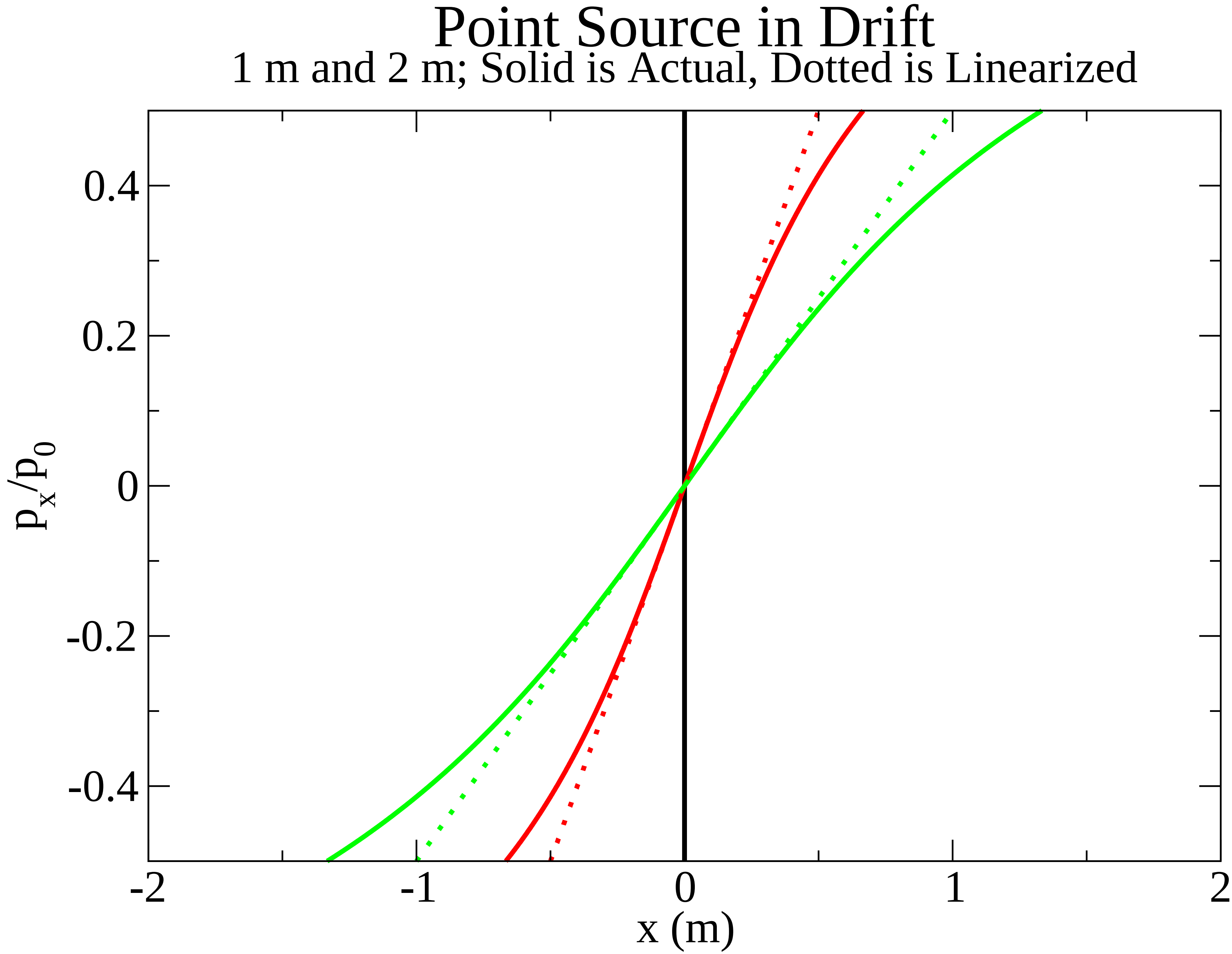
1. Block diagonalize $\mathbf{\Sigma}(s)J$; this gives you $A(s)$
   - Blocks are of form

$$\sigma(s) = \begin{bmatrix} 0 & \epsilon_k \\ -\epsilon_k & 0 \end{bmatrix}$$

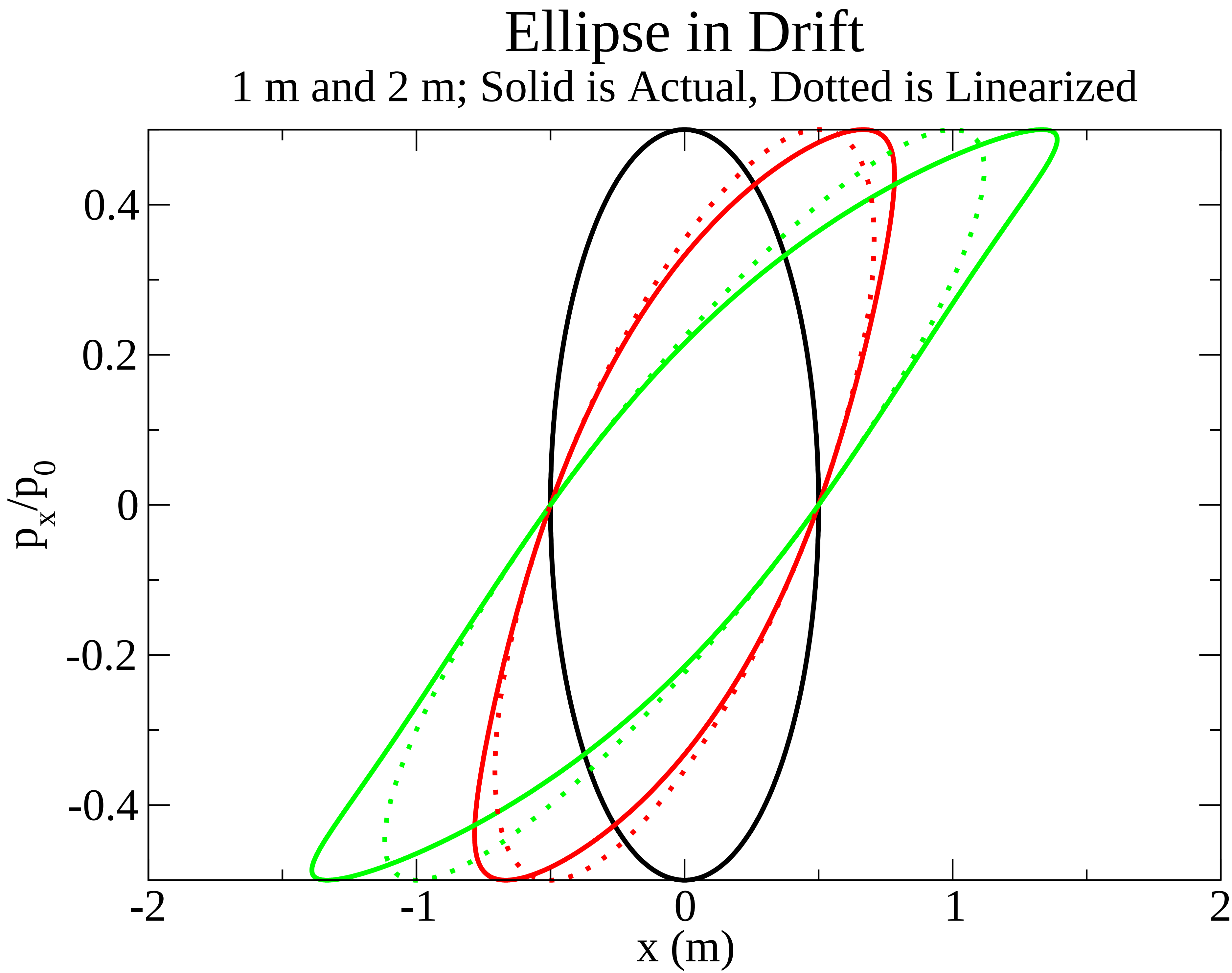
2. New equations are $\sigma(s')JR = R\sigma(s)J$
   - Exercise to the reader: ϵ_k are preserved!

Emittance Growth from Angles

- Constant energy point source
  - Initially a line in phase space
  - Becomes a twisted curve
- Start with upright ellipse
  - After drift, ellipse is distorted
- For given p, p_s decreases with increasing p_x
  - Larger p_x: particles take longer to traverse distance due to lower p_s
  - Thus, x change goes faster than linearly in p_x
- Alternative view: angles
  - $p_x/p = \sin\theta_x$
  - $dx/ds = \tan\theta_x$



Point Source in Drift
1 m and 2 m; Solid is Actual, Dotted is Linearized
0.4
0.2
0
-0.2
-0.4
-2
-1
0
1
2
x (m)
p_x/p_0


Ellipse in Drift
1 m and 2 m; Solid is Actual, Dotted is Linearized
0.4
0.2
0
-0.2
-0.4
-2
-1
0
1
2
x (m)
p_x/p_0

Compute Emittance Growth from Angles

- Compute emittance (use momenta scaled by referrence momentum):

$$x_1 = x_0 + \frac{p_0 L}{\sqrt{1-p_0^2}} \approx x_0 + p_0 L + \frac{1}{2}p_0^3 L + \frac{3}{8}p_0^5 L$$

$$\begin{aligned}\epsilon_1^2 = \langle x^2\rangle_1\langle p^2\rangle_1 - \langle xp\rangle_1^2 \approx \epsilon_0^2 + \langle p^2\rangle_0\langle xp^3\rangle_0 L - \langle p^4\rangle_0\langle xp\rangle_0 L \\ + \frac{3}{4}\langle p^2\rangle_0\langle xp^5\rangle_0 L - \frac{3}{4}\langle xp\rangle_0\langle p^6\rangle_0 L + \frac{1}{4}\langle p^2\rangle_0\langle p^6\rangle_0 L^2 - \frac{1}{4}\langle p^4\rangle_0^2 L^2\end{aligned}$$

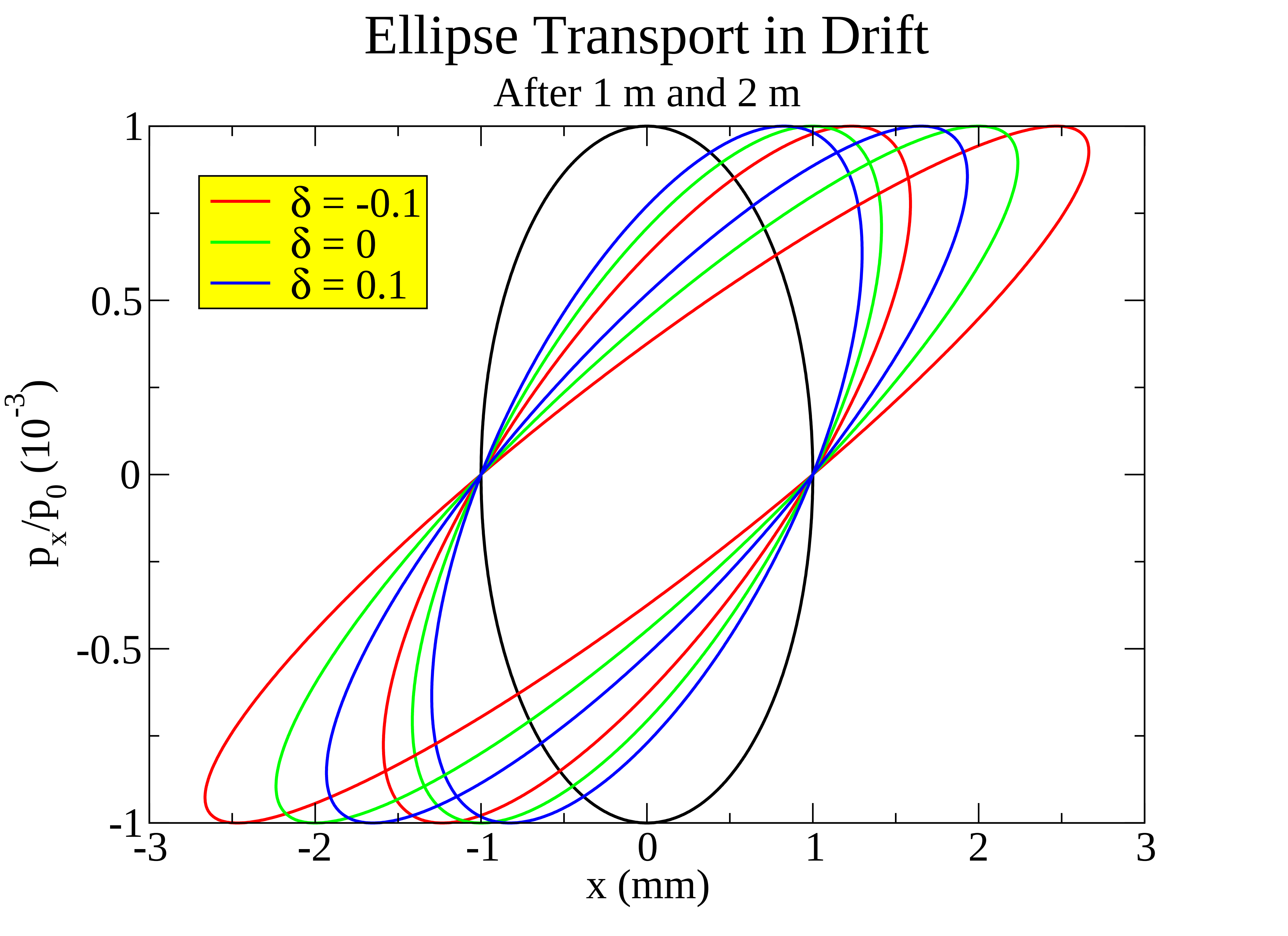
- Starting with upright Gaussian beam,

$$\epsilon_1^2 \approx \epsilon_0^2 + \frac{3}{2}\sigma_p^8 L^2$$

Emittance Growth from Energy Spread

- Larger total energy for given p_x
  - Arrives at destination faster
  - Less time to move transversely for given p_x
  - Less ellipse tilt
- Correlation between ellipse tilt and energy
  - This is a *third* order moment ($xp_x\delta$)
  - In second order moments, appears as extra terms: emittance growth
  - Upright Gaussian beam:

$$\epsilon_1^2 \approx \epsilon_0^2 + \sigma_p^4\sigma_\delta^2 L^2 + 3\left(\frac{L}{2\gamma^2\beta c}\right)^2 \sigma_\delta^6$$


Ellipse Transport in Drift
After 1 m and 2 m
δ = -0.1
δ = 0
δ = 0.1
p_x/p_0 (10^{-3})
x (mm)
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3

Conclusions

- Emittance comes from second order moment matrix
- Emittance is generally preserved only under linear transforms
- Drift is nonlinear
  - Comes both from angle and energy spread
- Nonlinearity leads to emittance growth