Errors calculations for the MARS yields

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1 Introduction

This document describes how the errors on the MARS yields are calculated, and how the errors on the total and relative differences between two yields (e.g the number of weighted muons from MARS 1507 and 1510) have been calculated. In addition the yields and errors from a comparison between two simulations for a 5 GeV proton beam in the ST2a configuration (a.k.a. ISS design), using MARS 1510, at BNL and CERN, is given as an example and for verification purpose.

2 Error on the weighted yield

The running sum of a set of N entries with values w_i (e.g. N muons from a MARS output file with respective weights w_1, w_2,...,w_N) can be written as follows:

$$S_j = \sum_{i=1}^{N} w_i^j \tag{1}$$

Therefore we have:

$$S_0 = N$$

$$S_1 = \sum_{i=1}^{N} w_i$$

and

$$S_2 = \sum_{i=1}^{N} w_i^2$$

The sample standard deviation of a set of N entries with values w_i is equal to:

$$S = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(w_i - \overline{w})^2} \tag{2}$$

And we have:

$$
\begin{aligned}
\sum_{i=1}^{N}(w_i - \overline{w})^2 &= \sum_{i=1}^{N}(w_i^2 - 2 \cdot w_i \cdot \overline{w} + \overline{w}^2) \\
&= \sum_{i=1}^{N} w_i^2 - 2 \cdot \sum_{i=1}^{N} w_i \cdot \overline{w} + N \cdot \overline{w}^2 \\
&= S_2 - 2 \cdot \sum_{i=1}^{N} w_i \cdot \frac{\sum_{i=1}^{N} w_i}{N} + N \cdot \frac{(\sum_{i=1}^{N} w_i)^2}{N^2} \\
&= S_2 - \frac{S_1^2}{S_0}
\end{aligned}
$$

Therefore S becomes:

$$
S = \Big(\frac{S_0 \cdot S_2 - S_1^2}{S_0 \cdot (S_0 - 1)}\Big)^{1/2} \tag{3}
$$

The variance of the sum of uncorrelated random variables is the sum of their variance:

$$
Var(\sum_{i=1}^{N} w_i) = \sum_{i=1}^{N} Var(w_i) \tag{4}
$$

If we assume Var(w_i) is the same for all the w_i and equal to S^2 we have:

$$
\begin{aligned}
Var(\sum_{i=1}^{N} w_i) &= N \cdot S^2 \\
&= \frac{S_0 \cdot S_2 - S_1^2}{S_0 - 1}
\end{aligned}
$$

So, as the error on the sum of the weight $E(\sum_{i=1}^{N} w_i)$ is the square root of the variance we have:

$$
E(\sum_{i=1}^{N} w_i) = \Big(\frac{S_0 \cdot S_2 - S_1^2}{S_0 - 1}\Big)^{1/2} = E
$$

This is in general smaller than $(\sum_{i=1}^{N} w_i)^{1/2}$ which is a rough estimator on the error on the sum of the weights and has been adopted as the standard way to calculate the error on a weighted yield (sum of indidual weights for a given set of N entries) by the IDS-NF collaboration [1].

3 Error on the absolute and relative difference between two yields

When comparing the yields between two different simulations (e.g. two different versions of MARS code or simulations performed at two different locations), the errors on the absolute and relative differences between the two simulations have been calculated as follows. Let's call Y_i the weigthed yield (sum of the individual weights from a given set of entries) of simulation i, DIFF the absolute difference Y_1 - Y_2 and FRAC the relative difference $\frac{Y_1-Y_2}{Y_1}$. The error on the absolute difference E(DIFF) can be approximated as:

$$E(DIFF) = \left(E_1^2 + E_2^2\right)^{1/2} \tag{5}$$

and the error on the relative difference E(FRAC) as:

$$E(FRAC) = |FRAC| \cdot \left(\frac{E_1^2}{Y_1^2} + \frac{E_2^2}{Y_2^2}\right)^{1/2} \tag{6}$$

3.1 Example of errors calculation

In Tables 1 and 2 the yields, absolute and relative differences and respective errors are reported for a comparison of two simulations of 10^5 protons on target with a 5 GeV beam using MARS 1510 at CERN and BNL. The comparison has been done for two distances z = 0 m and z = 50 m. This, with the intention to provide an example and make sure the errors calculation is performed correctly. Only particles for which the sum of the weights was above 100 are reported here.

In this case, Y_1 corresponds to the simulation from CERN and Y_2 to the simulation from BNL.

On Figures 1, 2 and 3, the yields, relative and absolute differences respectively are shown for the two locations, z = 0 and z = 50 m.

References

[1] J.S. Berg private communication

PID	Type	Y_1	E_1	Y_2	E_2	DIFF	E(DIFF)	FRAC	E(FRAC)
1	p	75134	319.7	75350	323.0	-216	454.4	-0.002	0.0060
2	n	56434	291.0	56612	296.7	-179	415.6	-0.003	0.0074
3	π^+	27629	187.6	27424	187.9	205	265.5	0.007	0.0096
4	π^-	24042	178.2	24395	180.5	-353	253.6	-0.015	0.0106
5	K^+	517	18.1	508	18.3	9	25.8	0.017	0.0499
6	K^-	-	-	-	-	-	-	-	-
7	μ^+	1177	63.1	1188	61.8	-11	88.3	-0.009	0.0750
8	μ^-	1199	61.4	988	56.2	211	83.3	0.176	0.0700
9	γ	22048	220.4	22032	209.2	16	303.9	0.000	0.0138
10	e^-	17714	122.7	18048	134.1	-333	181.7	-0.019	0.0103
11	e^+	19164	132.1	19854	143.0	-690	194.7	-0.036	0.0102
12	$\overline{p}$	-	-	-	-	-	-	-	
13	π^0	-	-	-	-	-	-	-	-
14	d	456	18.3	461	17.1	-5	25.1	-0.012	0.0550
15	t	-	-	-	-	-	-	-	-
16	^{3}He	-	-	-	-	-	-	-	-
17	^{4}He	-	-	-	-	-	-	-	-
18	ν_μ	2908	65.1	2792	63.5	115	90.9	0.040	0.0313
19	$\overline{\nu}_\mu$	1589	51.5	1465	48.4	125	70.7	0.079	0.0445
20	ν_e	1046	37.1	1088	39.4	-42	54.1	-0.040	0.0518
21	$\overline{\nu}_e$	-	-	-	-	-	-	-	-

Table 1: Yields and errors at z = 0

PID	Type	Y_1	E_1	Y_2	E_2	DIFF	E(DIFF)	FRAC	E(FRAC)
1	p	25654	229.4	25660	229.7	-5	324.7	-0.000	0.0130
2	n	-	-	-	-	-	-	-	-
3	π^+	2306	36.7	2474	40.3	-167	54.5	-0.073	0.0237
4	π^-	1636	30.3	1870	33.5	-234	45.2	-0.143	0.0278
5	K^+	-	-	-	-	-	-	-	-
6	K^-	-	-	-	-	-	-	-	-
7	μ^+	16706	151.9	16556	152.0	150	214.9	0.009	0.0129
8	μ^-	15135	145.4	15175	148.4	-41	207.7	-0.003	0.0137
9	γ	-	-	-	-	-	-	-	-
10	e^-	16154	118.8	16449	126.5	-294	173.6	-0.018	0.0107
11	e^+	17853	131.6	18133	133.9	-279	187.7	-0.016	0.0105
12	$\overline{p}$	-	-	-	-	-	-	-	
13	π^0	-	-	-	-	-	-	-	-
14	d	183	8.4	193	10.0	-9	13.0	-0.052	0.0711
15	t	-	-	-	-	-	-	-	-
16	^3He	-	-	-	-	-	-	-	-
17	^4He	-	-	-	-	-	-	-	-
18	ν_μ	128	8.9	163	12.4	-35	15.2	-0.277	0.1208
19	$\overline{\nu_\mu}$	138	11.6	134	10.5	4	15.7	0.027	0.1136
20	ν_e	-	-	-	-	-	-	-	-
21	$\overline{\nu_e}$	-	-	-	-	-	-	-	-

Table 2: Yields and errors at z = 50 m

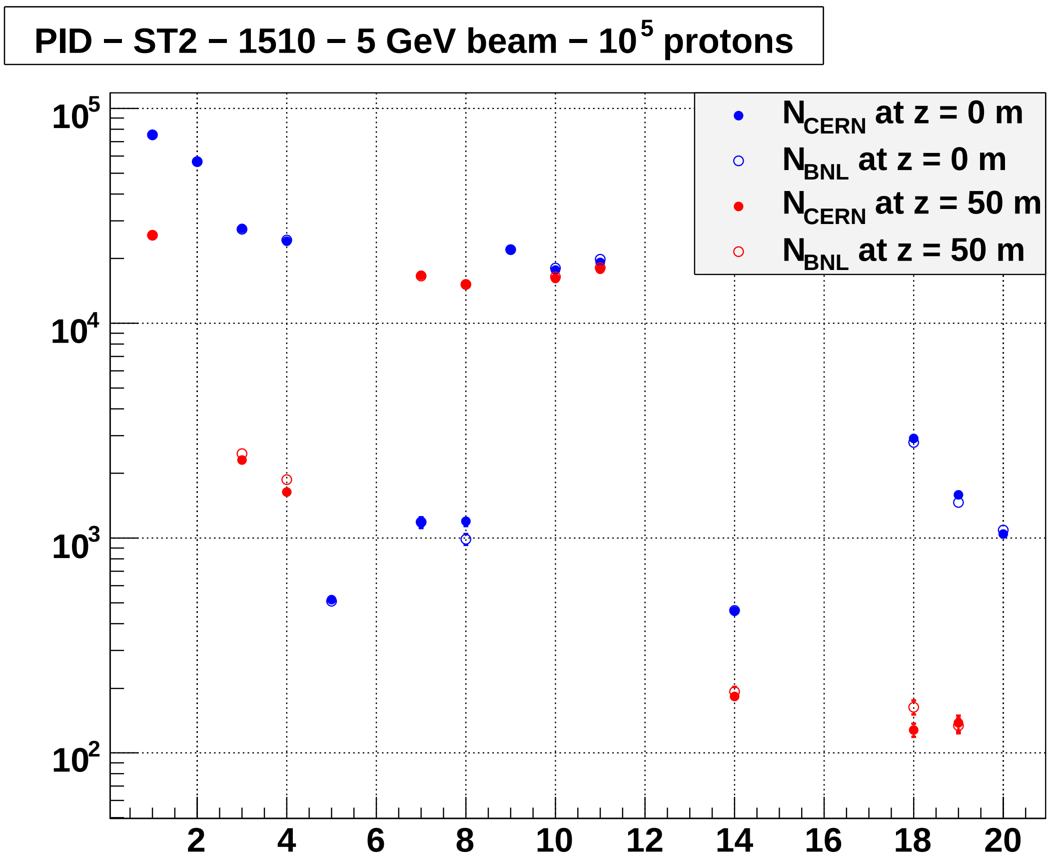

Figure 1: Yields from MARS at two locations z = 0 and z = 50 m.

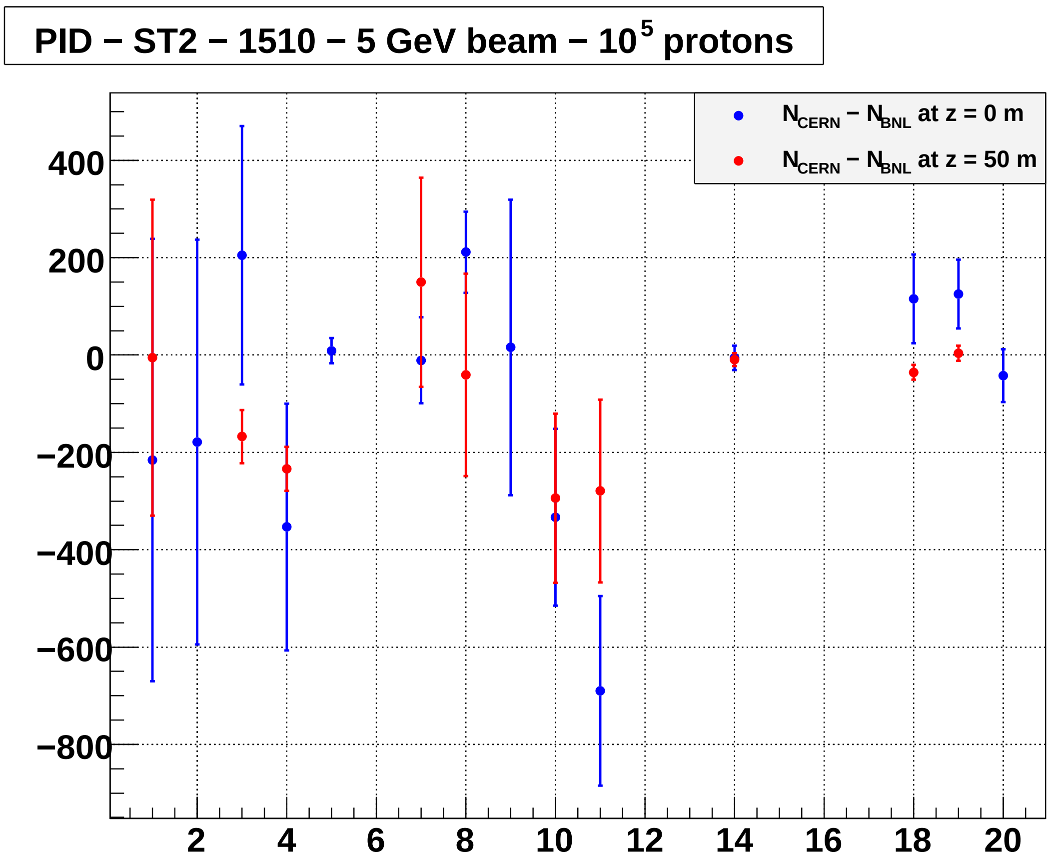

Figure 2: Absolute difference in the yield at two locations z = 0 and z = 50 m.

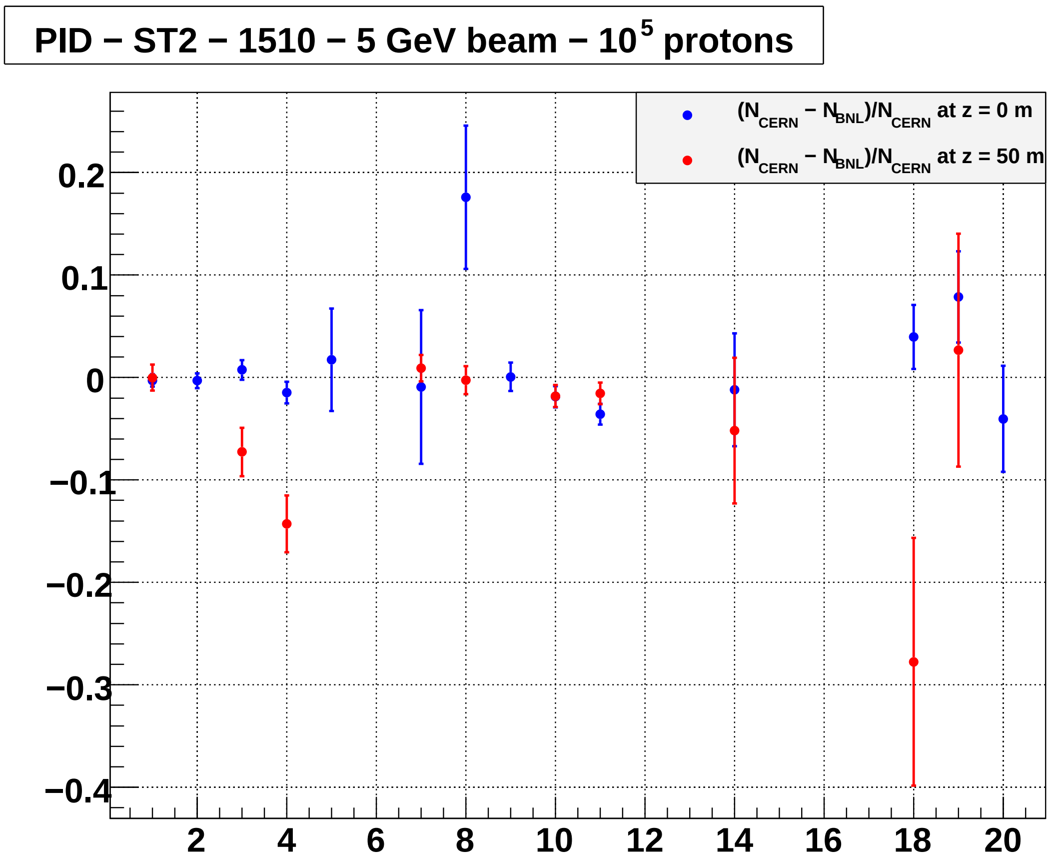

Figure 3: Relative difference in the yield at two locations z = 0 and z = 50 m.