Notes on the eigen-emittance concept
with application to MAP Front End design

Robert Ryne

Lawrence Berkeley National Laboratory

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What are eigen-emittances?

- Eigen-emittances are the generalization of canonical rms emittances to systems in which there are correlations between the phase planes of the beam distribution
  - in other words, generalization to systems in which the beam matrix (6x6 canonical 2nd moment matrix) contains nonzero terms outside the 2x2 diagonal blocks
- So, if you think that rms emittances are important quantities, then, in systems with correlations, the eigen-emittances are important quantities
- Notes:
  - when correlations are removed, eigen-emittances are the same as canonical rms emittances
  - beams born in longitudinal magnetic fields generally have coupling between the (x-px) and (y-py) phase planes.

Why are eigen-emittances important to accelerator and collider design?

- Beams are born with certain eigen-emittances
  - in the photo-injector of a light source or e+/e- collider
  - in the decay channel of the Front End of a muon collider
- The eigen-emittances don't change during transport in linear Hamiltonian systems
  - in these systems, whatever the beam is born with, that's what you are stuck with (you can, however, interchange the eigen-emittances)
- The eigen-emittances can change during transport in
  - nonlinear Hamiltonian systems
  - non-Hamiltonian systems
- Achieving high luminosity in muon colliders requires reducing the (initially large) eigen-emittances to small values via muon cooling

How to compute eigen-emittances in a beam dynamics code that uses canonical variables

- Compute the beam 2nd moment matrix, Σ
- Compute $J\Sigma$
- Compute the eigenvalues of $J\Sigma$
- The eigen-emittances are the modulii of the eigenvalues of $J\Sigma$

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

- If the code does not use canonical variables, then first you need to convert the data to canonical form, the follow the above procedure

- In summary, computing eigen-emittances is easy (assuming you can compute eigenvalues of a 6x6 matrix)
  - computing the transforming matrix that transforms Σ to Williamson normal form is more complicated.

Thoughts on MAP Front End design

- Muon beams are born in a decay channel containing a strong longitudinal B field
- The B field splits the "transverse" eigen-emittances, making one large and one small (with their product the same as the B=0 case)

$$\lambda^2_{\pm,solen} = \frac{1}{2}\left\{\varepsilon_x^2 + \varepsilon_y^2 + \sigma_{11}\sigma_{33}(\overline{q}B)^2 \pm \sqrt{-4\varepsilon_x^2\varepsilon_y^2 + \left[\varepsilon_x^2 + \varepsilon_y^2 + \sigma_{11}\sigma_{33}(\overline{q}B)^2\right]^2}\right\}$$

$$\overline{q} \equiv q / p_{sc}$$

- So, while large B helps to contain the muons, it also makes the job of cooling harder because it increases one of the eigen-emittances that the beam is born with
  - Does it matter in our parameter regime?
    - Example: B=2T, p_{sc}=240 MeV/c, unnormalized ε_x, ε_y=.036, s11=s33=(7.5cm)^2 → eigen-emittances are .044, .029

split is not significant in our case

- Summary: unless there is a significant change in our parameters, the decay channel B field does not significantly impact the eigen-emittances that the beam is born with

Thoughts on MAP Front End design, cont.

- Though the distinction between rms emittances and eigen-emittances is not significant in the decay channel, it might be elsewhere. We should compute it and use it when comparing different cooling concepts.
- Example code to compute eigen-emittances is contained in extra material at end of this presentation.

Additional material:
Details on eigen-emittances

What are eigen-emittances*#?

- Let z denote a 6-vector of canonical coordinates and moments, $z=(x,p_x,y,p_y,t,p_t)$
- Consider the matrix of 2nd moments, $Z_{ab}=<z_a,z_b>$
- For example, in a 1D system (2D phase space),

$$Z = \begin{pmatrix} < x^2 > & < xp_x > \\ < xp_x > & < p_x^2 > \end{pmatrix}$$

- Easy to verify that Z is diagonalized via a *symplectic congruency transformation*,

$$AZA^T = \varepsilon_{rms} I_{2x2} \qquad \text{where} \qquad A = \begin{pmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{pmatrix}$$

$$\alpha = -< xp_x > /\varepsilon_{rms}$$

$$\beta =< x^2 > /\varepsilon_{rms}$$

$$\gamma =< p_x^2 > /\varepsilon_{rms}$$

$$\varepsilon_{rms}^2 =< x^2 >< p_x^2 > - < xp_x >^2$$

*Material presented here is from Chapter 8, section 8.37, of the MaryLie manual and will eventually be found in Ch 26 of Alex Dragt's textbook, downloadable from http://www.physics.umd.edu/dsat/

#The earliest reference to "eigen-emittance" that I have found is A. J. Dragt et al., "Lie Algebraic Treatment of Linear and Nonlinear Beam Dynamics," Ann. Rev. Nucl. Part. Sci., Vol. 38, pp 455-496 (1988). The definition in terms of symplectic congruency transformations is found in the MaryLie manual.

See also: A. Dragt, R. Gluckstern, F. Neri, and G. Rangarajan, "Theory of Emittance Invariants", in Lecture Notes in Physics 343: Frontiers of Particle Beams; Observation, Diagnosis, and Correction, M. Month and S. Turner, Eds., Springer Verlag (1989); A. Dragt, F. Neri, and G. Rangarajan, "General moment invariants for linear Hamiltonian systems", Phys. Rev. A, p. 2572 (1992).

Eigen-emittances, cont.

- In the 1D case we found that: $AZA^T = \varepsilon_{rms} I_{2x2}$

- This is a special case of a famous theorem due to Williamson* that states#,
  - *for any real symmetric positive-definite 2nx2n matrix Z, there exists a symplectic matrix A such that*

$$AZA^T = \Lambda$$

$$\Lambda = \begin{pmatrix} \underline{\varepsilon}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \underline{\varepsilon}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{\varepsilon}_n & 0 \\ 0 & 0 & 0 & 0 & 0 & \underline{\varepsilon}_n \end{pmatrix}$$

$\underline{\varepsilon}_n$ *= modulii of the eigenvalues of JZ*

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

  - *Though A is not unique, it is the case that, for any matrix A that diagnolizes Z, the quantities (*$\underline{\varepsilon}_1$ *,* $\underline{\varepsilon}_2$ *,* $\underline{\varepsilon}_3$ *) are independent of A, up to a reordering*

- We call $\underline{\varepsilon}_n$ "eigen-emittances"
  - $\underline{\varepsilon}_n$ = rms emittances when Z is 2x2 block diagonal (i.e. if <xy>, <xp_y>,…= 0)

*J. Williamson, "On the Algebraic Problem Concerning the Normal Forms of Linear Dynamical Systems," Amer. J. Math. **58** (1) (1936), 141-163.

#See, e.g., the texbook by Maurice de Gosson, "Symplectic Geometry and Quantum Mechanics," 2006.

Eigen-emittances in 3D

- In the 1D case we found that

$$AZA^T = \underline{\varepsilon}_x I_x \qquad I_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- The analgous result in 3D is

$$AZA^T = \underline{\varepsilon}_1 I_1 + \underline{\varepsilon}_2 I_2 + \underline{\varepsilon}_3 I_3$$

$$I_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad I_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad I_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Note that the code MaryLie has built-in capabilities to compute eigen-emittances and related quantities.

Invariance of the eigen-emittances under linear symplectic transport

- Suppose that we have a distribution of particles and compute its 2nd moments, i.e. suppose we have computed the matrix given by $Z_{ab}=\langle z_a,z_b\rangle$
- There exists a symplectic matrix A that diagonizes Z via the congruency relation,

$$AZA^T = \underline{\varepsilon}_1 I_1 + \underline{\varepsilon}_2 I_2 + \underline{\varepsilon}_3 I_3$$

- Suppose we propagate particles to some final location via symplectic matrix M. Then the final 2nd moment matrix (e.g. at the end of the beamline) is

$$Z^f = MZM^T$$

- Now consider the matrix $B=AM^{-1}$:

$$BZ^f B^T = AM^{-1}MZM^T(M^T)^{-1}A^T = AZA^T = \underline{\varepsilon}_1 I_1 + \underline{\varepsilon}_2 I_2 + \underline{\varepsilon}_3 I_3$$

- Hence, the matrix B brings Z^f to normal form. Therefore we can also write:

$$BZ^f B^T = \underline{\varepsilon}_1^f I_1 + \underline{\varepsilon}_2^f I_2 + \underline{\varepsilon}_3^f I_3$$

- We conclude that

$$\underline{\varepsilon}_i^f = \underline{\varepsilon}_i \qquad \textit{i=(1,2,3), up to a reordering}$$

The eigen-emittances are invariant under linear symplectic transformations

Subroutine to compute eigen-emittances

```
! < coord(6,nraysp) is the particle array with canonical variables (x,px,y,py,t,pt) >
! < msk(nraysp) is a logical array used to mask off particles to be excluded >
      ngoodloc=count(msk(1:nraysp)) ! local # of "good" particles
      call MPI_ALLREDUCE(ngoodloc,ngood,1,mntgr,mpisum,lworld,ierror)
      den1=1./ngood   ! ngood is the global # of "good" particles
! compute beam centroid
      avloc(1)=sum(coord(1,1:nraysp),msk(1:nraysp))*den1
      avloc(2)=sum(coord(2,1:nraysp),msk(1:nraysp))*den1
      avloc(3)=sum(coord(3,1:nraysp),msk(1:nraysp))*den1
      avloc(4)=sum(coord(4,1:nraysp),msk(1:nraysp))*den1
      avloc(5)=sum(coord(5,1:nraysp),msk(1:nraysp))*den1
      avloc(6)=sum(coord(6,1:nraysp),msk(1:nraysp))*den1
      call MPI_ALLREDUCE(avloc,av,6,mreal,mpisum,lworld,ierror)
!   compute second-order moments
      vecm(1)= sum((coord(1,1:nraysp)-av(1))*(coord(1,1:nraysp)-av(1)),msk(1:nraysp))*den1
      vecm(2)= sum((coord(1,1:nraysp)-av(1))*(coord(2,1:nraysp)-av(2)),msk(1:nraysp))*den1
      vecm(3)= sum((coord(1,1:nraysp)-av(1))*(coord(3,1:nraysp)-av(3)),msk(1:nraysp))*den1
      vecm(4)= sum((coord(1,1:nraysp)-av(1))*(coord(4,1:nraysp)-av(4)),msk(1:nraysp))*den1
      vecm(5)= sum((coord(1,1:nraysp)-av(1))*(coord(5,1:nraysp)-av(5)),msk(1:nraysp))*den1
      vecm(6)= sum((coord(1,1:nraysp)-av(1))*(coord(6,1:nraysp)-av(6)),msk(1:nraysp))*den1
      vecm(7)= sum((coord(2,1:nraysp)-av(2))*(coord(2,1:nraysp)-av(2)),msk(1:nraysp))*den1
      vecm(8)= sum((coord(2,1:nraysp)-av(2))*(coord(3,1:nraysp)-av(3)),msk(1:nraysp))*den1
      vecm(9)= sum((coord(2,1:nraysp)-av(2))*(coord(4,1:nraysp)-av(4)),msk(1:nraysp))*den1
      vecm(10)=sum((coord(2,1:nraysp)-av(2))*(coord(5,1:nraysp)-av(5)),msk(1:nraysp))*den1
      vecm(11)=sum((coord(2,1:nraysp)-av(2))*(coord(6,1:nraysp)-av(6)),msk(1:nraysp))*den1
      vecm(12)=sum((coord(3,1:nraysp)-av(3))*(coord(3,1:nraysp)-av(3)),msk(1:nraysp))*den1
      vecm(13)=sum((coord(3,1:nraysp)-av(3))*(coord(4,1:nraysp)-av(4)),msk(1:nraysp))*den1
      vecm(14)=sum((coord(3,1:nraysp)-av(3))*(coord(5,1:nraysp)-av(5)),msk(1:nraysp))*den1
      vecm(15)=sum((coord(3,1:nraysp)-av(3))*(coord(6,1:nraysp)-av(6)),msk(1:nraysp))*den1
      vecm(16)=sum((coord(4,1:nraysp)-av(4))*(coord(4,1:nraysp)-av(4)),msk(1:nraysp))*den1
      vecm(17)=sum((coord(4,1:nraysp)-av(4))*(coord(5,1:nraysp)-av(5)),msk(1:nraysp))*den1
      vecm(18)=sum((coord(4,1:nraysp)-av(4))*(coord(6,1:nraysp)-av(6)),msk(1:nraysp))*den1
      vecm(19)=sum((coord(5,1:nraysp)-av(5))*(coord(5,1:nraysp)-av(5)),msk(1:nraysp))*den1
      vecm(20)=sum((coord(5,1:nraysp)-av(5))*(coord(6,1:nraysp)-av(6)),msk(1:nraysp))*den1
      vecm(21)=sum((coord(6,1:nraysp)-av(6))*(coord(6,1:nraysp)-av(6)),msk(1:nraysp))*den1
      call MPI_ALLREDUCE(vecm,gvecm,21,mreal,mpisum,lworld,ierror)
```

```
! compute the eigen-emittances:
      k=1
      do i=1,6
      do j=i,6
         gm(i,j)=gvecm(k)
         gm(j,i)=gvecm(k)
         k=k+1
      enddo
      enddo
      call mmult(jm,gm,xgm)
      call eig6(xgm,reval,aieval,revec,aievec)
      n=1
      do j=1,6
       if(aieval(j).gt.0.d0)then
        eig3(n)=aieval(j)
        n=n+1
       endif
      enddo
```