Emittance simulations

Pavel Snopok
IIT/Fermilab

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Emittance calculation

- $\Sigma = \mathrm{cov}(x, p_x, y, p_y, t, -E)$; $\varepsilon_{6D} = \frac{c}{m^3}\sqrt{\det \Sigma}$;
- $\Sigma_T = \mathrm{cov}(x, p_x, y, p_y)$; $\varepsilon_T = \frac{1}{m}\sqrt{\sqrt{\det \Sigma_T}}$;
- $\Sigma_L = \mathrm{cov}(t, -E)$; $\varepsilon_L = \frac{c}{m}\sqrt{\det \Sigma_L}$;
- $\lambda_1, \lambda_2, \lambda_3$ – eigen-values of $J\Sigma$, where J is a block diagonal matrix made up of three blocks $J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- $|\lambda_1|, |\lambda_2|, |\lambda_3|$ – eigen-emittances.
- I compare rms emittances with eigen-emittances for linear and nonlinear cases for drift and MICE Step IV.

Beam parameters

- $P_{ref} = 200$ MeV/c;
- gaussian beam,
- normalized longitudinal emittance 90 mm;
- normalized transverse emittance 6 mm;
- $\sigma_x = \sigma_y = 37$ mm;
- $\sigma_{p_x} = \sigma_{p_y} = 17$ MeV/c;
- $\sigma_{p_z} = 29$ MeV/c;
- $\sigma_t = 1.25$ ns;
- no dispersion.

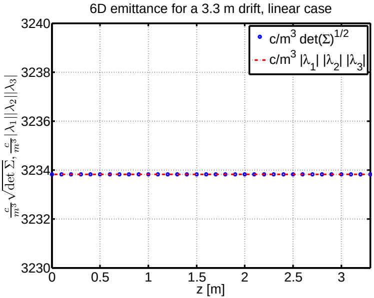
- Phase space is large, paraxial appoximation will not work.

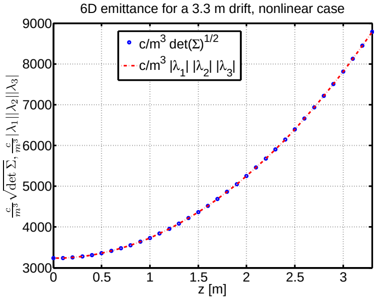
Linear vs nonlinear

- Our system can be described in terms of the flow f (propagator, transfer map): $\vec{z}_f = f(\vec{z}_i)$, where $\vec{z}_i$ – initial state of the system, $\vec{z}_f$ – final state (e.g. $\vec{z} = (x, x', y, y')$ for two dimensions).
- Most of the time we don't know the analytic expression for f, and we use numerical methods to obtain some approximation of f.
- Linear approximation: $\vec{z}_f = M\vec{z}_i$, where M is a matrix, (e.g., for one dimension $(x, x')_f = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} (x, x')_i^T$, where all m_{ij} are constants).
- Nonlinear approximation: there are different approaches to approximating f, in COSY that I used for calculations f is approximated by its Taylor polynomial of order n: $\vec{z}_f = T_n(f)(\vec{z}_i)$.

3.3 m drift

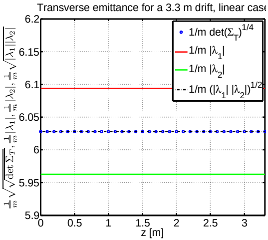
Drift, 6D emittance, linear vs nonlinear

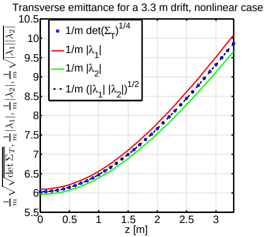




- ecalc9 uses: $\varepsilon_{6D} = \frac{c}{m^3}\sqrt{\det\Sigma}$.
- Equivalent to: $\frac{c}{m^3}|\lambda_1||\lambda_2||\lambda_3|$ in terms of eigen-emittances.
- Left: linear case; right: nonlinear case.
- Nonlinear case: emittance approximation based on second moment matrix Σ shows significant growth, while the phase space volume stays constant.

Drift, trans. emittance, linear vs nonlinear





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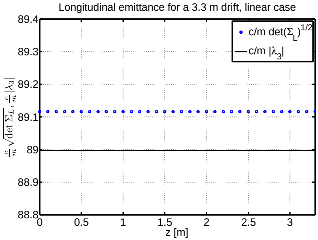
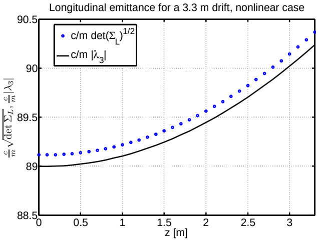
- Equivalent to: $\frac{1}{m}\sqrt{|\lambda_1||\lambda_2|}$ in terms of eigen-emittances.
- Left: linear case; right: nonlinear case.
- Two transverse eigen-emittances are different, but their geometric average is equivalent to the transverse emittance calculated by ecalc9.
- Nonlinear case: emittance growth, both for ε_T and $|\lambda_1|, |\lambda_2|$.

Drift, long. emittance, linear vs nonlinear

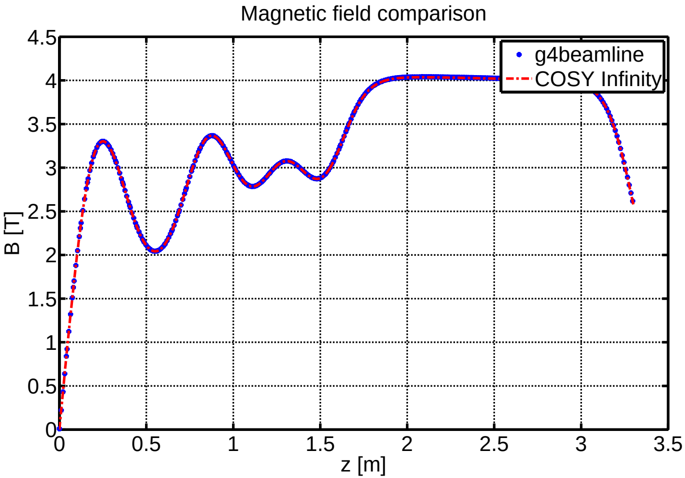




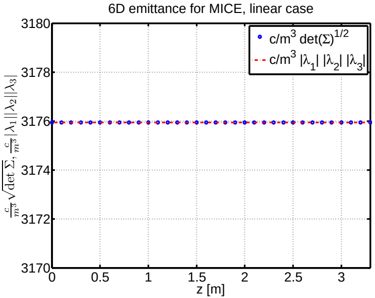
- ecalc9 uses: $\varepsilon_L = \frac{c}{m}\sqrt{\det \Sigma_L}$.
- Equivalent to: $\frac{c}{m}|\lambda_3|$ in terms of eigen-emittances.
- Left: linear case; right: nonlinear case.
- There is a slight difference due to the fact that ε_L uses only the part describing the longitudinal motion.
- Nonlinear case: emittance growth.

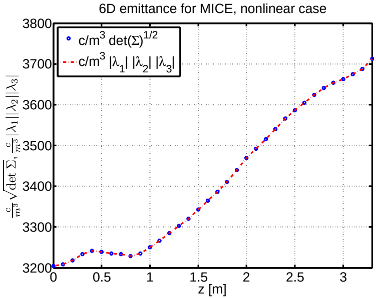
MICE Step IV geometry, no material

MICE Step IV magnetic field profile



MICE, 6D emittance, linear vs nonlinear





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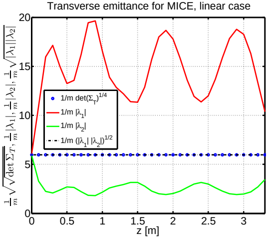

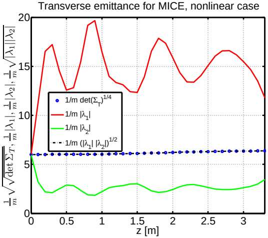

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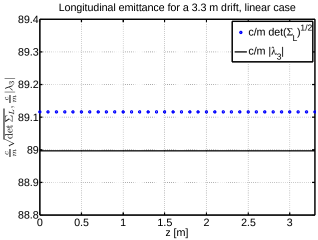
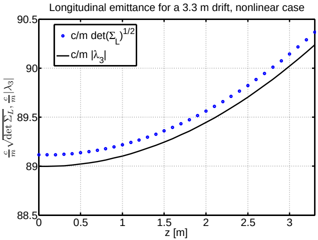
MICE, long. emittance, linear vs nonlinear





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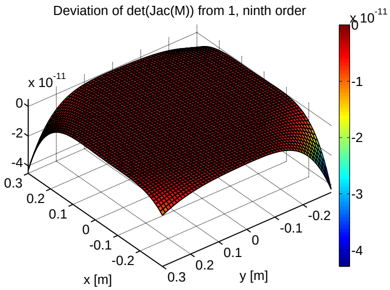
Phase space volume change in the nonlinear case (MICE Step IV)



- Phase space volume change can be determined by det($Jac(M)$).
- Calculation for the nonlinear case yields that the determinant is equal to 1 everywhere in the area of interest (based on the Taylor expansion of order 9).
- Picture shows the deviation of the determinant from 1 ($O(10^{-11})$).
- Phase space volume is constant.

Other ways to calculate emittance?

- Calculate phase space volume using Voronoi tesselation algorithms? – resource hungry
- Use det($Jac(M)$)? – how to include absorber material
- Higher moments?
- Do we need “nonlinear emittance”?