Least Squares Fitting of Ellipses

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In this section, we will detail the least squares method used to fit an ellipse to given points in the plane. In analytic geometry, the ellipse is defined as a collection of points (x, y) satisfying the following implicit equation [1]:

$$\tilde{A}x^2 + \tilde{B}xy + \tilde{C}y^2 + \tilde{D}x + \tilde{E}y = \tilde{F},$$

where $\tilde{F} \neq 0$ and $\tilde{B}^2 - 4\tilde{A}\tilde{C} < 0$. To simplify the following analysis, we normalize the above implicit form by dividing $\tilde{F}$ an both sides of the equality sign, which reduces to

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = 1. \tag{1}$$

Several new notations needs to be introduced to ease our discussion. For two vectors $\mathbf{s} = (s_1, s_2, \ldots, s_m)^{\mathrm{T}}$ and $\mathbf{t} = (t_1, t_2, \ldots, t_m)^{\mathrm{T}}$, the tensor product between them are defined as

$$\mathbf{s} \otimes \mathbf{t} = (s_1 t_1, s_2 t_2, \ldots, s_m t_m)^{\mathrm{T}}.$$

Assuming n measurements (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) are given, we define $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$, then the following cost function needs to be minimized

$$C(\boldsymbol{\beta}) = (\boldsymbol{X}\boldsymbol{\beta} - \mathbf{1})^{\mathrm{T}} (\boldsymbol{X}\boldsymbol{\beta} - \mathbf{1}),$$

where $\boldsymbol{X} = [\boldsymbol{x} \otimes \boldsymbol{x}, \boldsymbol{x} \otimes \boldsymbol{y}, \boldsymbol{y} \otimes \boldsymbol{y}, \boldsymbol{x}, \boldsymbol{y}]$ is a n-by-5 matrix, $\boldsymbol{\beta} = (A, B, C, D, E)^{\mathrm{T}}$ consists of the parameters to be determined, and $\mathbf{1}$ is a n-dimensional column vector with all 1's. Expand the matrix multiplication, we get

$$C(\boldsymbol{\beta}) = \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{X}^{\mathrm{T}} \boldsymbol{X} \boldsymbol{\beta} - 2\mathbf{1}^{\mathrm{T}} \boldsymbol{X} \boldsymbol{\beta} + n.$$

To minimize $C(\boldsymbol{\beta})$ it is requested that

$$\frac{\partial C(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = 2\boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{X}^{\mathrm{T}} \boldsymbol{X} - 2\mathbf{1}^{\mathrm{T}} \boldsymbol{X} = 0,$$

from which we get

$$\boldsymbol{\beta} = \left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{\mathrm{T}}\mathbf{1}.$$

The next step is to extract geometric parameters of the best-fitting ellipse from the algebraic equation (1). We first check the existence of a tilt, which is present only if the coefficient B in (1) is non-zero. If that was the case, we first need to eliminate the tilt of the ellipse. Denoting the tilt angle of the ellipse by θ, the following coordinate rotation transformation is employed

$$\begin{cases} x = \cos\theta x' - \sin\theta y' \\ y = \sin\theta x' + \cos\theta y' \end{cases}. \tag{2}$$

Substitute the above expressions into Eq. (1), we get

$$\left(Ac^2 + Bcs + Cs^2\right)x'^2 + \left(-2Acs + \left(c^2 - s^2\right)B + 2Ccs\right)x'y' +$$

$$\left(As^2 - Bcs + Cc^2\right)y'^2 + \left(Dc + Es\right)x' + \left(-Ds + Ec\right)y' + 1 = 0, \tag{3}$$

where $c = \cos\theta$ and $s = \sin\theta$. Let the term before xy to be zero, the following equation for θ is achieved

$$-2A\cos\theta\sin\theta + \left(\cos\theta^2 - \sin\theta^2\right)B + 2C\cos\theta\sin\theta = 0,$$

from which we know $\theta = \frac{1}{2}\arctan\left(\frac{b}{a-c}\right)$. Now the constants c and s are known, Eq. (3) is reduced to

$$A'x'^2 + C'y'^2 + D'x' + E'y' + 1 = 0, \tag{4}$$

where A', C', D' and E' are all known constants. The only remaining step for the ellipse fitting is to transform Eq. (4) into the following canonical form

$$\frac{\left(x' - x'_0\right)^2}{a^2} + \frac{\left(y' - y'_0\right)^2}{b^2} = 1, \tag{5}$$

in which (x'_0, y'_0) is the center of the ellipse in the rotated coordinate system, and a and b are the lengths of the semi-axes. Apply a square completion method to Eq. (3), we get

$$\frac{\left(x' + D'/\left(2A'\right)\right)^2}{\left(F'/A'\right)} + \frac{\left(x' + E'/\left(2C'\right)\right)^2}{\left(F'/C'\right)} = 1, \tag{6}$$

where $F' = -1 + (D'^2)/(4A') + (E'^2)/(4C')$. Compare Eqs. (5) and (6), it easy to notice

$$x'_0 = \frac{-D'}{2A'}, y'_0 = \frac{-E'}{2C'}, a = \sqrt{\frac{F'}{A'}}, b = \sqrt{\frac{F'}{C'}}.$$

Substitute the above expressions of x'_0 and y'_0 into Eq. (2), we get the coordinate of the ellipse center in the original coordinate system

$$\begin{cases} x_0 = -\cos\theta \frac{D'}{2A'} + \sin\theta \frac{E'}{2C'} \\ y_0 = -\sin\theta \frac{D'}{2A'} - \cos\theta \frac{E'}{2C'} \end{cases} .$$

References

[1] Cynthia Y. Young, Precalculus, John Wiley & Sons, 2010.