BNL 65611

CAP-220-MUON-98R

June 1998

Behavior of Conducting Solid or Liquid Jet Moving in Magnetic Field: 1) Paraxial; 2) Transverse; 3) Oblique

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Abstract. When a conductor moves through a nonuniform magnetic field, eddy currents flow that interact with the field to decelerate the conductor and perhaps change its trajectory, orientation and, if a liquid, shape. A rod of radius a = 1 cm and the density and electrical conductivity of melted gallium (γ = 6.1 g/cm^3, ρ = 26 μΩ cm) will decelerate 6.3 m/s in a 0.5 m ramp of paraxial field with a constant gradient g of 40 T/m (ΔB = 20 T). The deceleration is proportional to $a^2 g\Delta B/\gamma\rho$, independent of the velocity. The bar decelerates about twice as much in a 20 T, 0.5 m ramp of transverse field. A bar traveling at a shallow angle to such a field decelerates about 6.3 m/s. If the bar is 0.25 m long and moves at 20 m/s, it aligns with the field in ~10 ms, during which time it advances ~0.2 m.

Assumptions: In all cases the conductor is a cylindrical rod or rectangular bar of length l and density γ. The electrical resistivity ρ is sufficiently high that the field generated by eddy currents never rivals the ambient magnetic field. Currents respond instantaneously to voltages, according to Ohm's Law, without any inductive delay.

Case (1): Rod Moving in Paraxial Field: 1a) Linear Ramp; 1b) Field of Long Coil

In case (1) the conductor is a rod of radius a that is coaxial with an axisymmetric magnetic field and moves at a velocity v along their common z axis. Along the axis the field is purely axial: $\vec{B}(0,z) = B(z)\,\hat{z}$. Zero divergence of $\vec{B}$ implies that the radial component of field near the axis is, to first order, $B_r(r,z) = -r\,g(z)/2$, where $g(z)$ is the axial field gradient $dB(z)/dz$. A coaxial loop of radius r encircles a flux $\Phi(r,z)$ of $\pi r^2 B(z)$; the voltage induced around the loop is:

$$V(r,z,t) = -\frac{d\Phi(r,z)}{dt} = -\pi r^2 \frac{dB(z)}{dt} = -\pi r^2 \frac{dB(z)}{dz}\frac{dz}{dt} = -\pi r^2 g(z)\, v(t)\,.$$

The voltage induces a circumferential current density:

$$j(r,z,t) = \frac{V(r,z,t)}{2\pi\rho\, r} = -\frac{r}{2\rho} g(z)\, v(t)\,.$$

The current density interacts with the radial component of magnetic field to generate an axial force per unit volume:

$$f_z(r,z,t) = -j(r,z,t)\; B_r(r,z) \; = -r^2 g^2(z)\, v(t)/4\rho\,.$$

Note that this is proportional to $-v(t)$, and so the force always is one of deceleration.

The current density also interacts with the axial magnetic field to generate a radial force whose density is:

$$f_r(r,z,t) = j(r,z,t) B_z(z) = -r\,g(z)\, B_z(z)\, v(t)/2\rho\,.$$

In the paraxial case the magnitude of radial force density is much larger than the axial one, by the ratio B_z / B_r. This force compresses the conductor radially as it enters the field and tends radially to disperse a conducting jet as it exits. However, unless arcing maintains the circumferential path for current to flow, eddy currents should collapse, eliminating the forces on the jet—axial as well as radial. In any case, the radial boundary of the jet should remain within the confines of its enclosing tube of force. Any conductor whose boundary follows a tube of force sees no change in flux linkage, and hence develops no induced voltage, eddy current or force. Therefore, a liquid jet emerging from a field should expand no more rapidly than do the flux lines; its diameter should not double until the field has fallen by a factor of four.

Integration of the axial force density gives the total force $F(z,t)$ on the rod. Integrate with respect to r from 0 to a, and with respect to z from the rod's trailing edge $z_- \equiv z - l$ to its leading edge z. Division by the mass of the rod, $m = \pi\gamma a^2 l$, gives the acceleration, dv/dt:

$$F(z,t) = -\frac{\pi a^4}{8\rho}\left[G(z) - G(z_-)\right] v(t)\,; \quad \frac{dv}{dt} = -\frac{a^2}{8\gamma\rho l}\left[G(z) - G(z_-)\right] v(t)\,,$$

where G is the indefinite integral of $g^2(z)$. If $g(z)$ is constant, $G(z) - G(z_-) = g^2\, l$.

$$\frac{dv}{dt} = -\frac{a^2 g^2}{8\gamma\rho} v(t)\,; \quad \frac{dv}{v} = -\frac{a^2 g^2}{8\gamma\rho} dt \equiv -\frac{dt}{t_0}\,; \quad \ln(v) = -t/t_0\,; \quad v(t) = v_0\, e^{-t/t_0}\,,$$

where $t_0 = 8\gamma\rho / a^2 g^2$. The velocity decays exponentially with time.

For the more general case of a field gradient that depends on axial position, solve for the velocity as a function of position—the leading edge of the rod—rather than time.

$$\frac{1}{v}\frac{dv}{dt} = \frac{1}{v}\frac{dv}{dz}\frac{dz}{dt} = \frac{dv}{dz} = -\frac{a^2}{8\gamma\rho l}\left[G(z) - G(z_-)\right]\,; \quad v(z) = v_0 - \frac{a^2}{8\gamma\rho l}\int_0^z [G(z) - G(z_-)]dz.$$

For the case of constant gradient, this velocity reduction $\Delta v(z) \equiv v_0 - v(z)$ is $\Delta v(z) = a^2 g^2 z / 8\gamma\rho \;= a^2 g\, \Delta B / 8\gamma\rho$.

How big is this deceleration, in a typical target solenoid? Suppose that the rod is of gallium, with density and electrical resistivity appropriate to the liquid at its melting point of 30°C: γ=6.1x10^3 kg/m^3 [1] and ρ=26 μΩ–m [2,3]. Suppose that the rod climbs a 0.5 m long ramp with a slope of 40 T/m. For these values the reduction in velocity is 6.3 m/s for a rod of radius a=10^{-2} m. This is the minimum velocity for the rod to penetrate the field. Of course, were the rod in fact liquid, it might deform, because the body force density varies so greatly throughout its volume: radially as r^2 and axially as $g^2(z)$.

A more appropriate field profile is that of a current sheet solenoid. If its radius is a_1 and its ends are at zero and $+\infty$, the field $B(z)$ is $B_0\left[1 + z/(a_1^{\,2} + z^2)^{1/2}\right]$, where B_0, the field at the mouth of the solenoid, is exactly half the peak field. The field gradient $g(z)$ is $B_0\left[a_1^2/(a_1^2+z^2)^{3/2}\right]$, with maximum value B_0/a_1 at the solenoid mouth, $z=0$. The indefinite integral $G(z)\equiv\int g^2(z)dz$ is:

$$G(z)=B_0^{\,2}a_1^{\,4}\int\frac{dz}{\left(a_1^{\,2}+z^2\right)^3}=\frac{B_0^{\,2}}{8}\left[\frac{3}{a_1}\tan^{-1}\frac{z}{a_1}+\frac{\left(5a_1^{\,2}+3z^2\right)z}{\left(a_1^{\,2}+z^2\right)^2}\right].$$

To predict the rod's cumulative deceleration, $\Delta v(z)\equiv v(-\infty)-v(z)$, evaluate $\left[G(z)-G(z_-)\right]$ and integrate with respect to z from $-\infty$ to z:

$$\Delta v(z)=\frac{a^2B_0^{\,2}}{64\lambda\rho l}\left[\frac{3z}{a_1}\tan^{-1}\frac{z}{a_1}-\frac{3z_-}{a}\tan^{-1}\frac{z_-}{a_1}+\frac{a_1^{\,2}}{a_1^{\,2}+z_-^{\,2}}-\frac{a_1^{\,2}}{a_1^{\,2}+z^2}+\frac{3\pi l}{2a_1}\right]$$

$$\Delta v(\infty)=\frac{3\pi a^2B_0^{\,2}}{64a_1\gamma\rho}.$$

A gallium rod as above decelerates 4.6 m/s in going from $-\infty$ to $+\infty$ along the axis of a semi-infinite solenoid of 0.2 m radius and 20 T peak field. The deceleration scales as the square of the solenoid field and the rod radius, and inversely as the solenoid radius and the rod density and electrical resistivity.

To observe the effect of a magnetic field on a liquid jet a particularly convenient metal is gallium A hot summer day is enough to melt the pure metal, while a eutectic alloy with 25% indium melts at 16 °C [4]. A ternary alloy with 22% indium and 16% tin, melts at 11 °C [4]. Its viscosity, 1.894 centipoise at 53 °C [5] for example, is only double that of water. It is relatively nontoxic, and, unlike mercury, its vapor pressure is very low [6]. Unfortunately, gallium is fairly expensive—99.99% purity gallium is $930 for 200g from one source [7].

Case (2): Deceleration of Conductor by Transverse Field with Constant Gradient

Case (2) calculates the force on a conductor of length l that moves along its axis through a magnetic field perpendicular to that axis. Define the y and z axes to be those of the velocity v and field B, respectively. The velocity depends on time t or, equivalently, position y; the latter is the more informative. For simplicity demand that the field depend only on y, and that it be linear over the full distance of travel: $B(y) = B_0 + g\,y$, where g, the field gradient dB/dy, is constant. Let the cross section be rectangular, with transverse dimension $2a << l$ in the x direction. The other transverse dimension—parallel to the field—has no influence on the rate of deceleration.

When the center of the of the bar is at y (and its leading and trailing edges at $y \pm l/2$, respectively), the longitudinal cross section of the bar encloses a flux $\Phi(y) = 2a\,l\,(B_0 + g\,y)$. This flux induces, around the boundary of the bar, a voltage:

$$V(t) = -\frac{d\Phi(y)}{dt} = -\frac{d\Phi(y)}{dy}\frac{dy}{dt} = -2a\,l\,g\,v(t)\,; \quad V(y) = -2a\,l\,g\,v(y)\,.$$

The voltage induces a current density which at the longitudinal surfaces $x = \pm a$ of the bar is approximately $j(\pm a, y) \approx \pm V(y)/2l\,\rho \approx \mp a\,g\,v(y)/\rho$.

Note that the voltage and current density depend not on the magnitude of the field but only on its gradient. Motion of the bar through the magnetic field produces a Lorentz force $F = \pm e\,v \times B$ on each proton and electron in the bar, causing some electrons to migrate to the surface $x = -a$; this polarizes the bar (in the x direction, perpendicular to its velocity and to the magnetic field). For a current to flow along the bar, however, it must have a gradient of voltage, and hence of magnetic field, along its length.

To predict the current density at any point (x, y), with $|x| < a$ and $|y| < l\,x/2a$, calculate the longitudinal current density at the surface of a geometrically similar but smaller bar, centered at (0,0) and of length and width $l' = u\,l$ and $2a' = 2u\,a$, where $0 \le u \le 1$. The current density at the longitudinal surface of the small bar should approximate the longitudinal current density at the same point in the interior of the large bar: $j_y(x) = \mp g\,v(y)\,x/\rho$. Note that the current density is proportional to the distance x from the (y,z) midplane of the bar. Since the longitudinal current density is antisymmetric in x, and interacts with a magnetic field that is independent of x, the sidewise force $dF_x = j_y \times B$ integrates to zero.

The assumption that current flows in rectangular paths of similar shape implies that the current that flows longitudinally at $x = \pm u\,a$ flows transversely at $y = \pm l'/2 = \pm u\,l/2 = \pm l\,x/2a$. The transverse current flows a distance $2x$. The positive transverse current at $+l\,x/2a$ and the equal but opposite current at $-l\,x/2a$ interact with fields that differ by $g\,l\,x/a$. The transverse force per increment of x therefore is:

$$dF_y / dx = 2x\; j_y(x)\,\Delta B(x) = -[2x][g\,v(y)\,x/\rho][g\,l\,x/a] = -2l\,g^2 v(y)\,x^3/\rho a\,.$$

Integrating with from 0 to a gives the total retarding force, $F_y \; = -a^3 l\,g^2\, v(y)\;/2\rho$.

This force, divided by the mass of the bar, $m = 2al\gamma$, is the acceleration dv/dt. More enlightening is the equation from section (1) that relates the force to dv/dy:

$$\frac{1}{v}\frac{F}{m} = \frac{dv}{dy}; \quad \frac{dv}{dy} = -\frac{a^3 l\,g^2/2\rho}{2a\,l\,\gamma} = -\frac{a^2\,g^2}{4\gamma\rho}; \quad \Delta v \equiv v_0 - v = \frac{a^2 g^2}{4\gamma\rho}\,y\,.$$

As with paraxial motion, the deceleration rate scales with the square of the bar thickness and the field gradient, and inversely with its density and electrical resistivity.

Upon negotiating a transverse field that ramps by 20 T over a distance of 0.5 m (40 T/m gradient), a gallium bar 0.02 m thick would decelerate about 12.6 m/s. To avoid so much deceleration, use either several smaller jets in parallel, or else a metal or alloy of higher density or higher electrical resistivity. Candidates elements, sequenced by melting point, are mercury (at. no. 80), at 13.6 g/cm^3 and 96 μΩ cm at 20 °C [2,8]; indium (at. no. 49), at 7.3 g/cm^3 and 33 μΩ cm at 156 K [2,9]; tin (at. no. 50), at 7.3 g/cm^3 and 48 μΩ cm at 232 °C [2,9]; bismuth (at. no. 83), at 9.8 g/cm^3 and 130.2 μΩ cm at 271 °C [2]; thallium (at. no. 81), at 11.8 g/cm3 and 73 μΩ cm at 303 K [2,8]; and lead (at. no. 82), at 11.4 g/cm^3 and 95 μΩ cm at 327 °C [2]. Such jets should penetrate adequately into an intense magnetic field, avoiding the likelihood that a non-conducting compound will decompose from the energy deposition from the intense proton beam. Unfortunately, mercury and thallium are highly toxic. Bismuth, under proton bombardment, transmutes to polonium, whose bad reputation rivals that of plutonium. Bismuth content may render unsuitable many alloys, such as Indalloy #15, otherwise attractive for its high density of 9.3 g/cm^3 and low melting point of 38-43 °C [4].

Case (3): Deceleration and Rotation of Conductor Moving at Small Angle to Field

The previous two sections prepare the way for a more general geometry in which a square bar aligned with the z axis moves along that axis at a modest angle with respect to a field of constant gradient $g \equiv dB_z / dz$. The bar is of length l and thickness $2a$ in both transverse dimensions. The longitudinal field is $B_z = B_{0z} + g\,z$; $\nabla \cdot \vec{B} = 0$, with $B_x \equiv 0$ everywhere, constrains the transverse component to be $B_y = B_{0y} - g\,y$.

Approximate the induced current flow by a set of square loops that girdle the bar transversely throughout its length and at all "radii" $|y| < a$. Each loop encloses a flux $\Phi(z) = 4y^2\,(B_{0z} + g\,z)$. This flux induces around the loop a voltage:

$$V(t) = -\frac{d\Phi(z)}{dt} = -\frac{d\Phi(z)}{dz}\frac{dz}{dt} = -4y^2 g\,v(t)\,; \quad V(z) = -4y^2 g\,v(z)\,.$$

The voltage induces a transverse current density at y of approximately $-V(z)/8y\rho$ $\approx y\,g\,v(z)/2\rho$. For the typical pion capture system of cases (1) and (2), with $g = 40$ T/m, $v = 20$ m/s and $\rho = 0.26\,\mu\Omega\,$m, the current density at the surface ($y = a = 10$ mm) is 15 A/mm^2.

This transverse current interacts with the z component of field to pinch the bar in the x and y directions. The body force density σ_y is $j_x B_z = y\,g\,v(z)\,B_z / 2\rho$. Integration with respect to y from 0 to a gives the hydrostatic pressure at the center of the bar as $g\,v(z)\,B_z\,a^2 / 4\rho$. For the pion capture system above, and with $B_z =$ 20 T, this pressure is ~1.5 MPa, or ~220 psi—about half a dozen times that from a water tap.

The current density j_x in one pair of sides of each loop also interacts with the y component of field to decelerate the bar and align it with the field. The deceleration force density on the sides is $f_z = -j_x\,[B_y(y) - B_y(-y)] = -[y g\,v(z)/2\rho][-2yg]$ $= y^2 g^2\,v(z)/\rho$. The force on a symmetric pair of current sheets of width $|2y|$ and length l is $2l\,y^3 g^2 v(z)/\rho$. Integrating with respect to y from 0 to a gives the total force as $l\,a^4 g^2\,v(z)/2\rho$.

Since the mass m of the bar is $4\gamma l a^2$, the decelerating force per unit mass is $a^2 g^2 v(z) / 8\gamma\rho$. Integrating as in section (2) gives $\Delta v(z) \equiv v_0 - v(z) = a^2 g^2 z / 8\gamma\rho$. This deceleration is half that of case (2), because the current flows in square loops with a ratio of area to perimeter, and hence of current density, that is only half that of the highly-elongated rectangles of case (2). For the pion capture system parameters as above, the deceleration is ~6.3 m/s. Note that this is the same as for the paraxial case. The result should be reasonably accurate even though the bar rotates and loses its alignment with its axis, so long as the misalignment never approaches 45º (i.e., the field in the direction of the velocity of the bar always is bigger than the field perpendicular to the velocity).

Again, the deceleration is inversely proportional to the density and electrical resistivity, and directly proportional to the square of the transverse dimension and the gradient of the field traversed. For an experiment to reveal the magnetic interaction, one wants low density and electrical resistivity. Candidate elements are sodium, at 0.97 g/cm^3 and 4.9 μΩ cm at 27 °C [10], potassium, at 0.86 g/cm^3 and 7.5 μΩ cm at 27 °C [10], or lithium, at 0.53 g/cm^3 and 13.4 μΩ cm at 127 °C [10] and 25 μΩ cm just above its melting point of 180 °C [8]. One also wants large diameter and field gradient. Unfortunately, too low a velocity may allow gravity to dominate the motion. Similarly, too large a diameter may allow turbulence to dominate. Fortunately, the field gradient can be high without the field itself being high, if the gradient is sufficiently localized.

The current density also tends to align the bar with the magnetic field. The torque density, measured about the $y = 0$ midplane of the bar, is $-y\, j_x\, B_y(y) = -[y][y\, g\, v(z) / 2\rho][B_{0y} - g\, y]$. The total torque on a current sheet of width $|2y|$ and length l is $-l|y|y^2 g\, v(z)[B_{0y} - g\, y] / \rho$. Integrating with respect to y from $-a$ to a gives the total torque as $\tau_x(z) = -l\, a^4 g\, B_{0y} v(z) / 2\rho$. The moment of inertia about the x axis of the bar is $I_x = m(l^2 + 4a^2)/12 \approx ml^2/12 \approx \gamma l^3 a^2 / 3$. Therefore the angular acceleration of the bar is $\ddot{\theta} = \tau_x / I_x \approx -3a^2 g\, B_{0y} v(z) / 2\gamma\rho l^2$, where θ is the angle between the bar and the total field.

B_{0y}, the transverse component of field, is $B_z \tan\theta$; if θ is modest, then $B_{0y} \approx B_z\,\theta$. The equation for angular acceleration then becomes that of simple harmonic motion, $\ddot{\theta} = -\omega^2\theta$, where $\omega \approx (a/l)\sqrt{3g\,B_{0z}\,v/2\gamma\rho}$. For the above pion capture system, with l = 25 cm, this angular frequency is ~160 radians per second; the bar aligns with the field in $\pi/2\omega \approx 10$ ms. If the bar retained its velocity of 20 m/s throughout this time, it will have moved about 0.2 m. A jet of 1 cm instead of 2 cm diameter should align in about 20 ms, in a distance of about 0.4 m. The tendency of the jet to align in a distance comparable to its length may be very useful: one need not inject the jet along the axis of the proton beam; one can use the magnetic field to make the jet collinear with the beam.

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