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Estimated perturbations of the axial motion of a liquid-metal jet entering a strong magnetic field

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One of the important issues that needs to be addressed when considering the injection of a fast, liquid-metal jet target into a strong magnetic field are the perturbations of the jet motion and shape due to the presence of large field gradients at the entrance of the solenoid [1]. Recently Lebedev [2] developed simple arguments claiming that the magnetic "pinch" effect caused by $\mathbf{I} \times \mathbf{B}$ forces on the induced eddy-currents, when a mercury jet enters such a 20T solenoid, will tend to produce large incremental velocities. The approach we follow here is similar, but the results and conclusions are much less pessimistic, and much more in line with the initial estimates [1] and with recent calculations [3]. Here (as in [2]) we will only consider a jet collinear with the solenoid.

We will use the so called ballistic or impulse approximation in which forces, accelerations and velocities are calculated while displacements are assumed to be negligible during the duration of the interaction. This is the same type of assumption made by others dealing with this problem when they postulate a constant radius for the jet. Our interaction time is the length of the fringe field region, divided by the velocity of the jet. Thus, for displacements to be small, the velocities generated in the frame of reference of the incoming jet must be small compared to the jet velocity. If the velocities resulting from the calculation fulfill this condition, then one can rely quantitatively on the results.

For these estimates we shall assume a liquid metal jet of radius r_o and velocity V_z centered on the axis z of a solenoid of radius R. Since $r_o << R$ the field component $B_z(z)$ is nearly independent of the radius for radii from 0 to r_o . The radial component $B_r(r,z)$ is obtained from $\nabla \mathbf{B} = 0$:

$$B_r = -\tfrac{1}{2}\, r\, \delta B_z / \delta z \qquad (1)$$

as can be easily verified by setting to zero the total flux out of a pill-box-shaped volume around the z axis.

There will be an electric field E in a frame of reference moving with the jet due to the changing magnetic flux:

$$\int E\, dl = -1/c \quad \delta/\delta t \int B_z\, ds$$

for any closed loop in a plane perpendicular to z. If that closed loop is a circle of radius r, centered on the axis we get an azimuthal field:

$$E = -r/2c \;\; \delta B_z/\delta t$$

If the conductivity of the medium is, σ we get a eddy current density $j = \sigma E$:

$$j = -\sigma r/2c \;\; \delta B_z/\delta t$$

But the field is changing in time because the jet is moving at a velocity V_z :

$$j = -\sigma r/2c \;\; \delta B_z/\delta z \;\; \delta z/\delta t = -\sigma r/2c \;\; V_z \;\; \delta B_z/\delta z \qquad (2)$$

This azimuthal current density interacting with the two field components B_r and B_z will then produce two force density components $f_z = -j\, B_r /c$ and $f_r = j\, B_z /c$ respectively, which can be written using (2) and (1):

$$f_z = -\sigma r/2c\; V_z \;\; \delta B_z/\delta z \;\times\; \tfrac{1}{2}\, r \delta B_z / \delta z \;\times 1/c \;=\; -\sigma r^2/4c^2 \;\; V_z \;\; (\delta B_z/\delta z)^2 \qquad (3)$$

$$f_r = -\sigma r/2c^2 \;\; V_z \;\; B_z\, \delta B_z/\delta z \qquad (4)$$

The longitudinal component f_z slows down the jet as it enters the magnet (and also as it exits, since the field gradient appears squared in (3)).

The inward radial component f_r is responsible for the "magnetic pinch effect" which has caused recent concerns [2]. At the exit of the magnet, these forces are outward since the gradient changes sign in (4). These outward forces may well disrupt the jet, but here we will only consider the strong inward forces present during the penetration of the fringe field at the entrance to the magnet.

Since the liquid is assumed to be incompressible, radial inward velocities can not be developed without setting up much larger (for our geometry) longitudinal velocities. One simple way to evaluate this situation is through the following steps:

a) To a good approximation at each point the radial inward force-density, $f_r = -\sigma r/2c^2 \; V_z \; B_z \, \delta B_z/\delta z$ (4), is compensated by an equal and opposite radial hydrostatic pressure gradient $\delta P(z,r) / \delta r$.
b) Integrating this gradient from r to r_o one writes the pressure $P(z,r)$.
c) The longitudinal pressure gradient, $\delta P(z,r) / \delta z$, is then used to calculate the resulting longitudinal, forces (f'_z), accelerations and velocities.

$$P(z,r) = \int_{r_o}^{r} (\delta P(z,r) / \delta r) \, dr = \sigma/2c^2 \; V_z \; B_z \, \delta B_z/\delta z \int_{r_o}^{r} r \, dr = \sigma \, (r_o^2 - r^2)/4c^2 \; V_z \; B_z \, \delta B_z/\delta z \qquad (5)$$

$$f_z' = -\delta P(z,r) / \delta z = -\sigma \, (r_o^2 - r^2)/4c^2 \; V_z \, [B_z \, \delta^2 B_z/\delta z^2 + (\delta B_z/\delta z)^2] \qquad (6)$$

Now we can write the equation of motion for a given volume element as:

$$dv_z/dt = (f_z + f_z')/\rho \qquad (7)$$

where v_z is the velocity in the moving reference frame of the incoming jet, and ρ is the density.

To compute the values of v_z by using (3) and (6) we need first to know the field derivatives which depend on the detailed solenoid coil geometry and currents. Here, as an approximation, we shall simply use the formulas for the field on the axis of a semi-infinite solenoid of radius R, and we shall locate the origin of the z-axis at the edge of the solenoid:

$$B_z = [1 + z / (z^2 + R^2)^{1/2}] \, B_{oz} /2 \qquad (8)$$

$$\delta B_z/\delta z = [\, 1/ (z^2 + R^2)^{1/2} - z^2 / (z^2 + R^2)^{3/2}] \, B_{oz} /2 \qquad (9)$$

$$\delta^2 B_z/\delta z^2 = [\, -3z / (z^2 + R^2)^{3/2} + 3 z^3 / (z^2 + R^2)^{5/2}] \, B_{oz} /2 \qquad (10)$$

Where B_{oz} is the field inside the solenoid. As mentioned before, the on-axis field and its derivatives are good approximation for the entire volume of the jet since $r_o << R$.

Using (9) and (10) in (3) and (6) we get:

$$f_z = -\sigma r^2/4c^2 \; V_z \; [\, 1/ (z^2 + R^2)^{1/2} - z^2 / (z^2 + R^2)^{3/2}]^2 \, B_{oz}^2 /4 \qquad (11)$$

and

$$f_z' = -\sigma \, (r_o^2 - r^2)/4c^2 \; V_z \; \{[1 + z / (z^2 + R^2)^{1/2}] \, [\, -3z / (z^2 + R^2)^{3/2} + 3 z^3 / (z^2 + R^2)^{5/2}] + [\, 1/ (z^2 + R^2)^{1/2} - z^2 / (z^2 + R^2)^{3/2}]^2\} B_{oz}^2 /4 \qquad (12)$$

Numerical results were obtained with Excel spread sheet calculations using (7), (11) and (12). Before discussing these results, lets see what happens with (11) and (12) at z=0, the entrance to the solenoid, where the fringe-field effects are large:

$$f_{(z=0)} = -\sigma r^2/4c^2 \ V_z \ \ 1/R^2 \ B_{oz}^2/4 \qquad (13)$$

$$f'_{(z=0)} = -\sigma (r_o^2 - r^2)/4c^2 \ V_z \ 1/R^2 \ B_{oz}^2/4 \qquad (14)$$

The first force density component is zero at the center where the second one is largest and vice versa. The maximum values are identical. The second component is the one due to the magnetic pinch effect. We can already see that nothing dramatic will happen due to this second term beyond the effects of the first one, which is the one that has been considered all along [1]. On the contrary, the overall effect of the second term should be smaller because the values of f_z' are large in the center, i.e. over a much smaller area.

To get an idea of the order of magnitude of the fringe-field effects we can evaluate an acceleration, e.g. from (13):

$dv_z/dt = f_z / \rho$

with

σ = 9.1E15 1/s
r = 1 cm
c = 3E10 cm/s
V_z = 2000 cm/s
R = 10 cm
B_{oz} = 200,000 gauss
ρ = 13.6 g/cm^3

We get:

$(dv_z/dt)_{z=0,r=ro} = 9.25 \times 10^3 \ cm/s^2$

A rough value for the interaction time for our example is ~10 ms (the typical interaction length ~2R = 20 cm , where the fringe field variation is pronounced, divided by the 2000 cm/s jet velocity). This yields a velocity variation of $9.25 \times 10^3 \ cm/s^2 \times 0.01s = 92.5$ cm/s . This estimate is somewhat larger than the maximum 55 cm/s value obtained in the detailed calculations, but in any case much smaller than the jet velocity, thus justifying the use of the impulse approximation for this case.

The calculation was set up by selecting points every 2 cm from –50 cm to +50 cm on the z-axis. For each point, the field and field derivatives where computed as outlined above, and the accelerations and velocity increments where calculated for six radii from 0 to the jet radius.

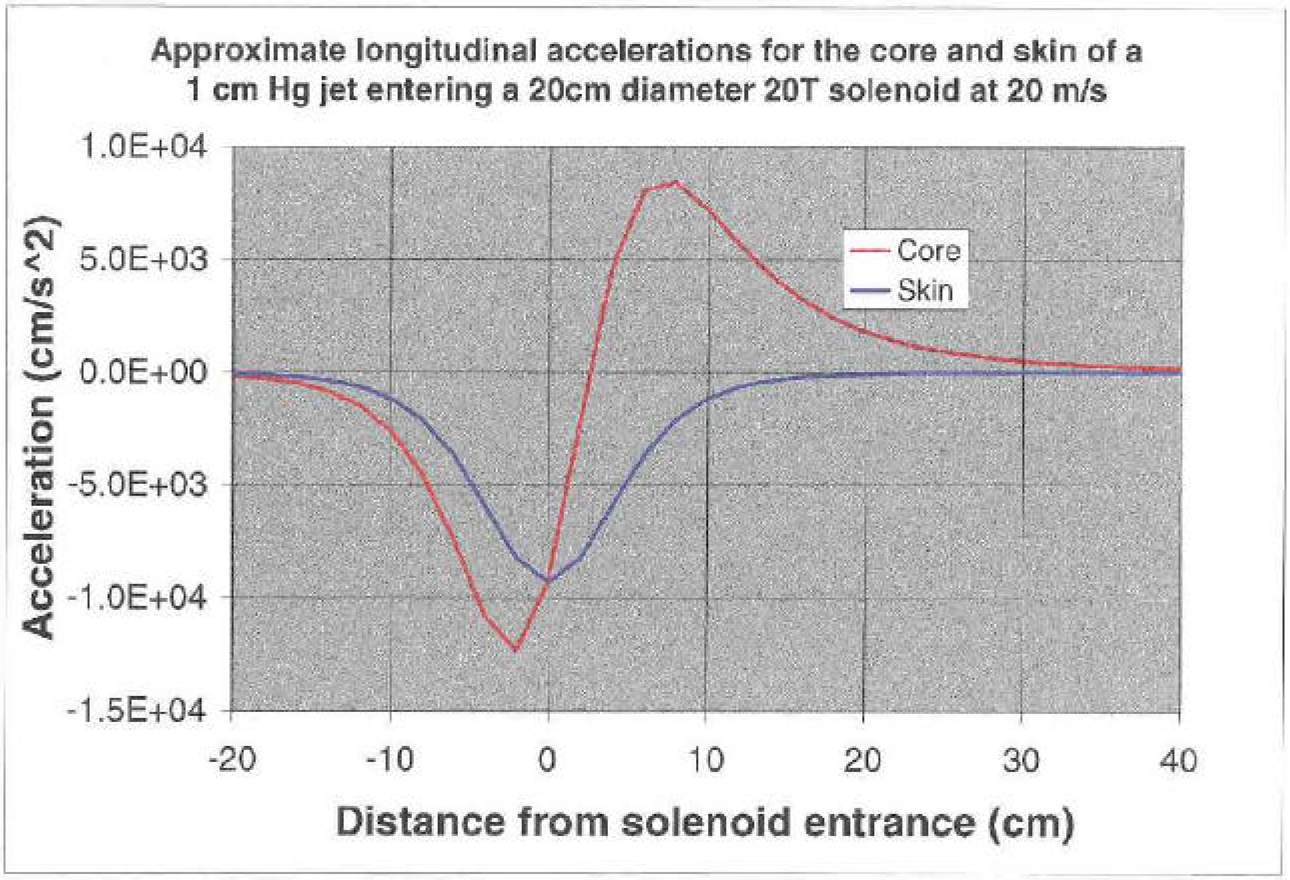

Fig. 1

The result for the accelerations at the core ($r = 0$) and at the surface ($r = r_o$) of the jet are shown in fig. 1. The "magnetic pinch effect", i.e. the interaction of the eddy currents with the z-component of the field is entirely responsible for the acceleration at the core and the interaction of the eddy currents with the r-component of the fringe field causes the acceleration at the surface. The surface is always decelerated, while the core is decelerated at first and then accelerated after entering the solenoid. This last effect is the result of the radial pressure tending to "squeeze out" material towards both the front and the back of the jet.

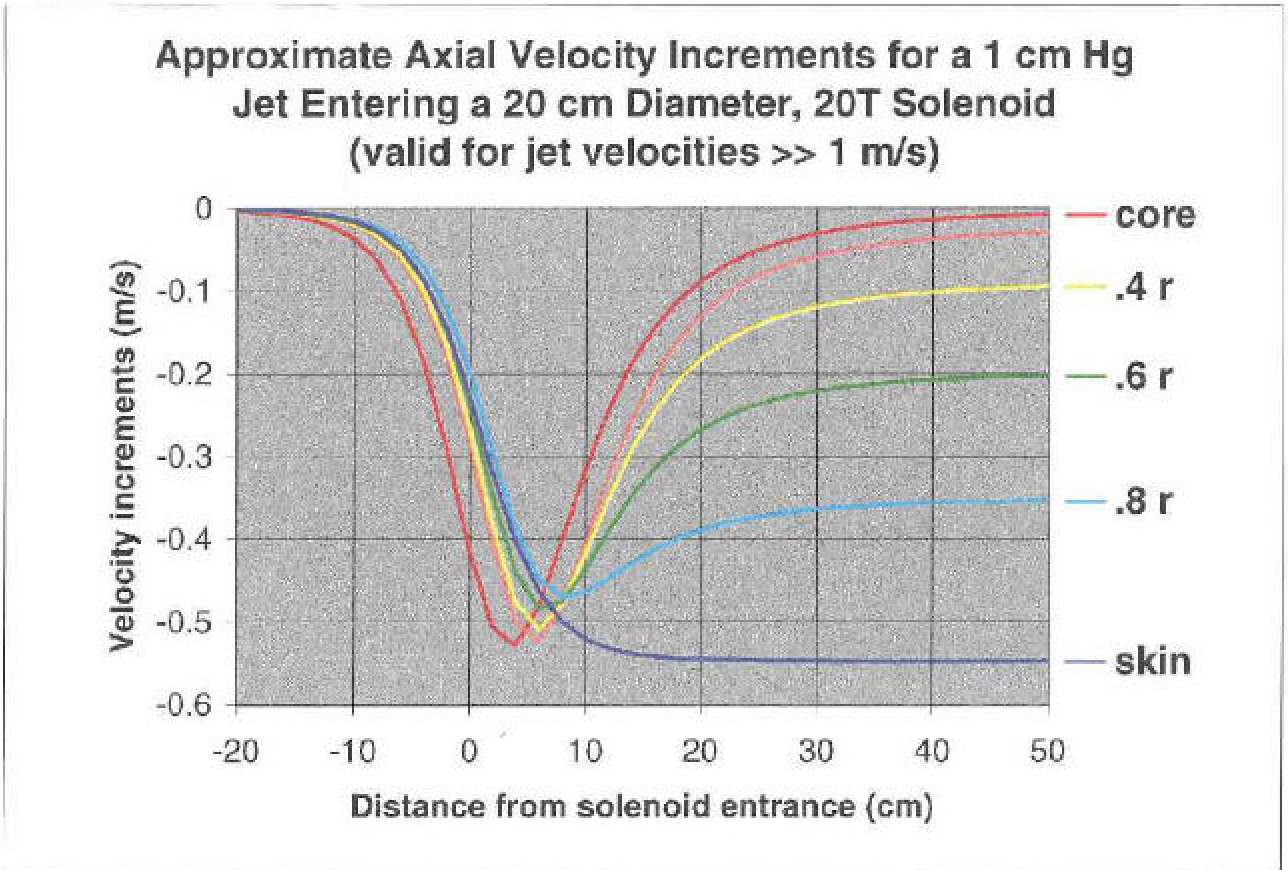

Fig. 2

Figure 2 shows the velocity increments at the core and five radii as functions of the distance from the entrance to the solenoid. The values shown in these curves are independent of the initial jet velocity. This must be so, because the forces are proportional to that velocity and the interaction time is inversely proportional to it. The impulse approximations, which is very good at 20 m/s will become less accurate for much smaller velocities.

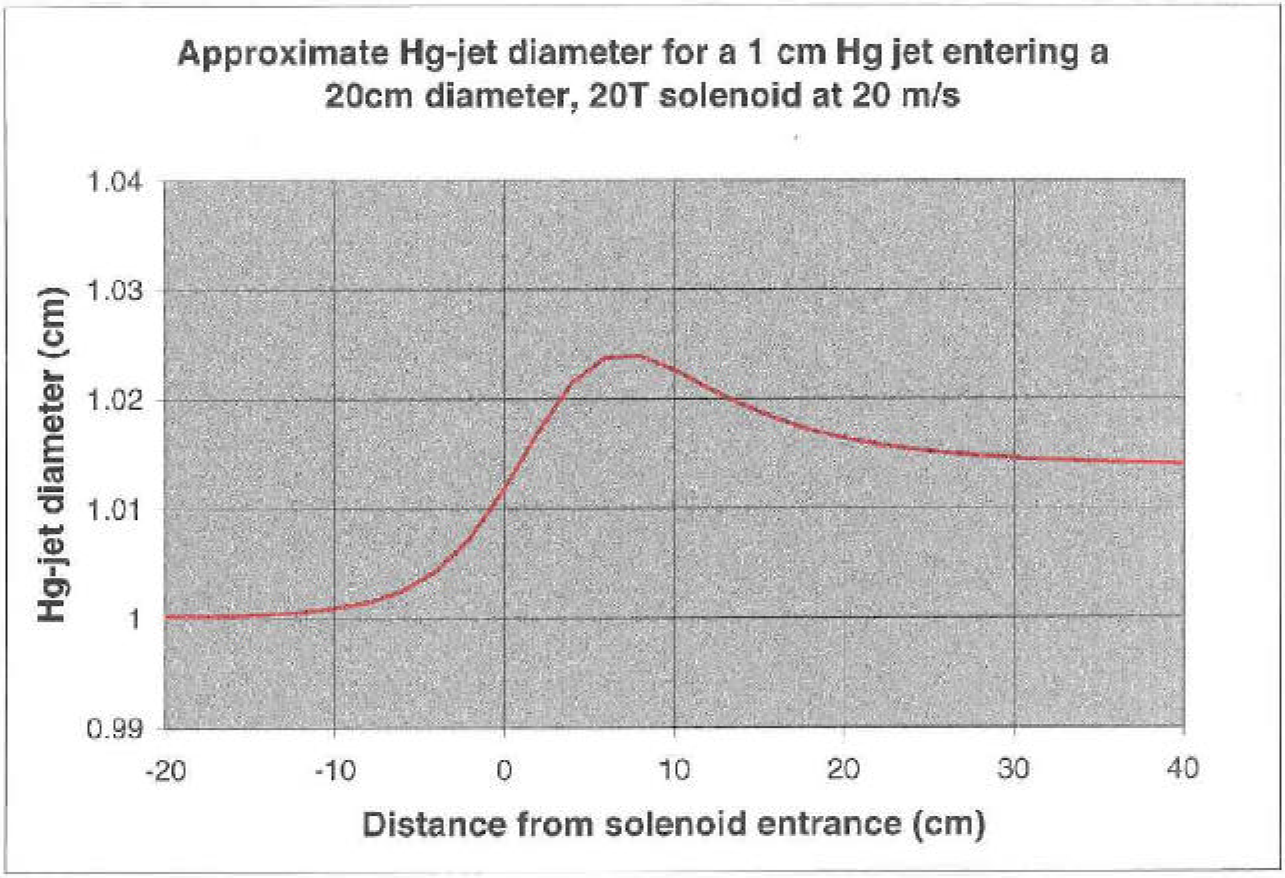

Fig. 3

The radius of the jet can be evaluated, using average jet velocities computed at each point from the previous six values, properly weighted with their respective cross-sectional areas, and using the fact that the volume-per-second of liquid crossing a plane must be constant for steady-state flow. The result is shown in figure 3. We see that in this case there is only a 2.3% variation of the radius, further confirming the validity of the approximations.

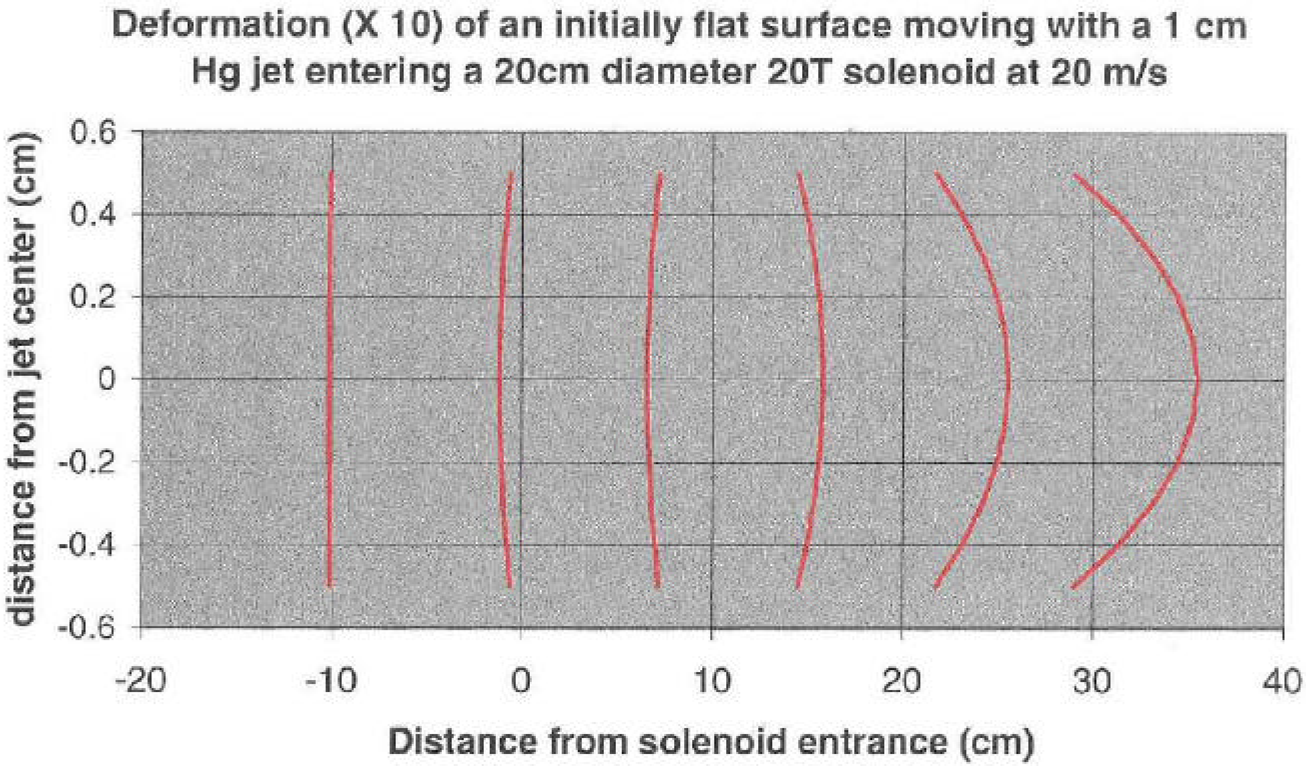

Fig. 4

The data of figure 2 were used to compute the deformation of an initially flat surface perpendicular to the jet, as the jet enters the solenoid. The result is shown in figure 4 where the retardation and deformations are exaggerated by a factor 10 to make them more visible. The grid lines to the right of each surface indicate the position were the surfaces would be in the absence of magnetic effects.

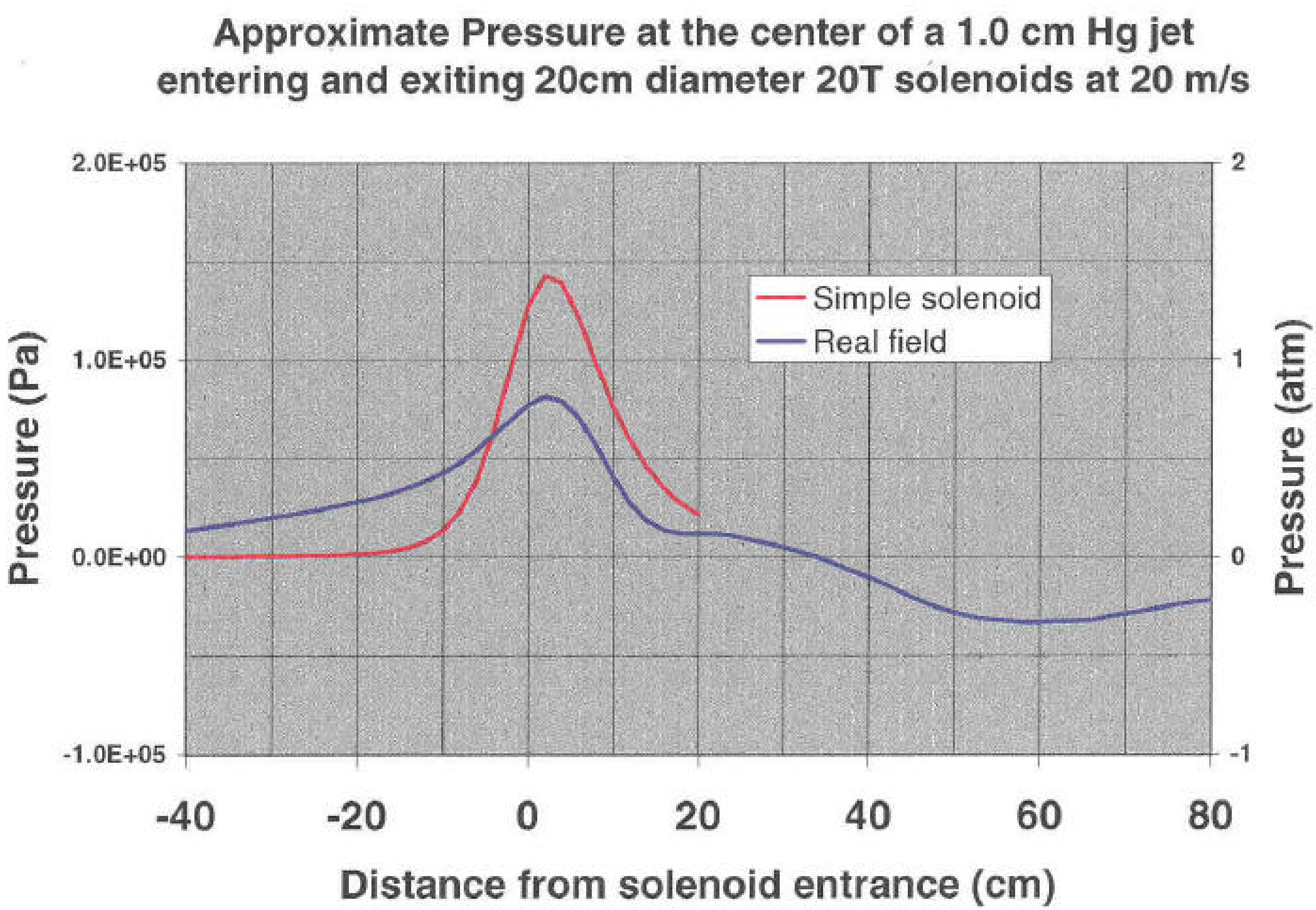

Fig. 5

Finally, fig. 5 shows pressures at the core of the jet calculated using equation (5) , first for the same simple solenoid used for the rest of the study and then for realistic fields [6] calculated for the standard capture coil package surrounding the target [7]. As expected, the magnetic-pinch-effect pressure spike at the entrance of the coil is considerably reduced for the real field as compared to the simple solenoid because of the larger radius and longitudinal distribution of components of the capture coil package. The other effects shown in figs 1-4 will be similarly reduced if the real field is used instead of the simple solenoid.

At the exit of the 20T region the field decreases more gradually due to the presence of the matching solenoids [7]. The negative pressure excursion is therefore much smaller and more spread out . The most negative value is -0.3 atmospheres. All these values were obtained assuming 0-pressure in field-free regions. If instead the jet operates in a 1 atmosphere environment, the minimum pressure will be +0.7 atmospheres. In this case there will then be no cavitation and no jet disruption due to the inverse pinch effect.

Discussion and conclusions

The computed velocity increments are relatively small for the planned 20 m/s, 1 cm diameter mercury jet entering a 20T solenoid. They will be further reduced by viscosity, which was not considered here, and by magnetic damping as recently described by McDonald [4]. Only jets collinear with the solenoid were considered here. Since the effects encountered are so small, it seems safe to conclude that nothing drastic will occur in the real, more complicated case where the jet is slightly inclined with respect to the solenoid axis.

The approximation of considering only longitudinal motion in the equation of motion (7) is justified by the aspect ratio of the system, namely a ~20cm interaction length compared to a 5mm jet radius. Any collective radial motion , as the one caused by magnetic squeezing, will be amplified roughly by that ratio when converted to longitudinal motions. Momentum and kinetic energies associated with the radial component can thus be neglected to good approximation. Another way to state the approximation is to say that, in the calculation, the slight radial accelerations are allowed to proceed unimpeded by inertia.

Our approximation is similar to the approximation used in the traditional derivation of Bernoulli's formula for the velocity of a stream of liquid emanating from a hole in the side of a tank. There kinetic energy in the stream is simply traded for potential energy in the tank. The slow and complicated motion of the liquid in the tank can be ignored to good approximation due to the small value of the ratio of the jet radius to the tank radius. Instead of uniform gravitational forces, in our problem we have variable inward radial forces, but they change over distances large compared to the jet radius.

Viscosity, magnetic damping [4] and inertial effects associated with radial motion have all been neglected, and, when incorporated, will all tend to reduce the velocity variations and jet deformation. Instabilities caused by surface tension (which was also ignored) may, on the other hand, be triggered by the traversal of the fringe field. The time constant for the development of these instabilities is probably long enough to avoid problems, and magnetic damping will also help considerably [4].

A rigorous solution of this problem requires a detailed magnetohydrodynamic calculation which is beyond the scope of this note. Compared to the complexities normally encountered by specialists in that field, the present problem should be simple. We have cylindrical symmetry, an incompressible liquid of constant density, viscosity and conductivity, and a simple magnetic field.

The Excel spreadsheet used to generate the figures is available [5] and can be used to evaluate different dimensions, velocities and target materials. For that purpose, input parameters have been placed in highlighted and labeled cells at the top of the spreadsheet. The upper and lower scale limits for some charts may need to be reset manually. The titles of the charts are automatically adjusted to correspond to the input parameters. Note that the approximations made will not be valid unless the radius of the jet is much smaller than the radius of the solenoid, and the incremental velocities obtained are much smaller than the initial jet velocity. These conditions are fairly well satisfied in the example presented in this note.

References

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