B(z) from Equations with Only One or Two Adjustable Currents

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Figure 1 plots on-axis field profiles generated by an equation, analytically differentiable to arbitrary order, each of whose terms predicts the on-axis field from a component coil in a Target Magnet. For simplicity, coils typically are modeled by current sheets. Comparison of the blue and black curves of Fig. reveals that even a coil as radially thick as Superconducting Coil #1 can be modeled accurately by a single current sheet at its mean radius.

For simplicity, each of the seven designs of Fig. 1 employs no more than three current elements, all those downstream of $z \approx 5.4$ m being consolidated into a single current sheet. Design #1 uses only two elements—a main coil and a current sheet extending from 2.5 m all the way to 20 m. Coil parameters are tabulated below. B_∞ is the field that would be generated by a coil of identical current per unit length but of infinite length.

Design Number	#1	#2	#3	#4	#5	#6	#7
Desired field @ z = L = 5 m [T]	2.5	2.5	1.5	2.0	2.5	3.0	3.5
Inner radius a_1 of coil #1 [m]	1.20	1.20	N/A	N/A	N/A	N/A	N/A
Outer radius a_2 of coil #1 [m]	2.08	2.08	N/A	N/A	N/A	N/A	N/A
Mean radius a_0 of coil #1 [m]	1.64	1.64	1.64	1.64	1.64	1.64	1.64
a_0 of downstream coils [m]	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Upstream end of coil #1 [m]	−2.35	−2.35	−2.35	−2.35	−2.35	−2.35	−2.35
Downstream end of coil #1 [m]	1.34	1.34	1.34	1.34	1.34	1.34	1.34
Upstream end of coil #2 [m]	2.50	2.50	2.50	2.50	2.50	2.50	2.50
Downstream end of coil #2 [m]	20.0	5.42	5.42	5.42	5.42	5.42	5.42
Upstream end of coil #3 [m]	N/A	5.427	5.427	5.427	5.427	5.427	5.427
Downstream end of coil #3 [m]	N/A	20.0	20.0	20.0	20.0	20.0	20.0
$j\lambda$ of thick-walled coil [A/mm^2]	18.01	18.06	N/A	N/A	N/A	N/A	N/A
B_∞ of upstream current sheet [T]	N/A	N/A	20.07	20.07	20.07	20.07	20.07
B_∞ of middle current sheet [T]	N/A	1.684	0.623	1.153	1.684	2.214	2.744
B_∞ of downstream sheet [T]	2.5	2.5	1.5	2.0	2.5	3.0	3.5

To satisfy constraints on the field magnitude B(z) and slope dB/dz at z =−0.5 m and z = L, one can adjust the energization level of coils. Only coil currents—not coil dimensions—are variables, so that solving for variables requires no iteration. Furthermore, for the designs of Fig. 1 only the interior coil is used to satisfy the requirement that $B(L) = B_{min}$. The upstream coil generates exactly 15 T without help for the downstream coils, which contribute only a few hundredth of a tesla at z = −0.5 m. The downstream current sheet is energized so as to generate exactly B_{min} deep in the interior of a coil of infinite length.

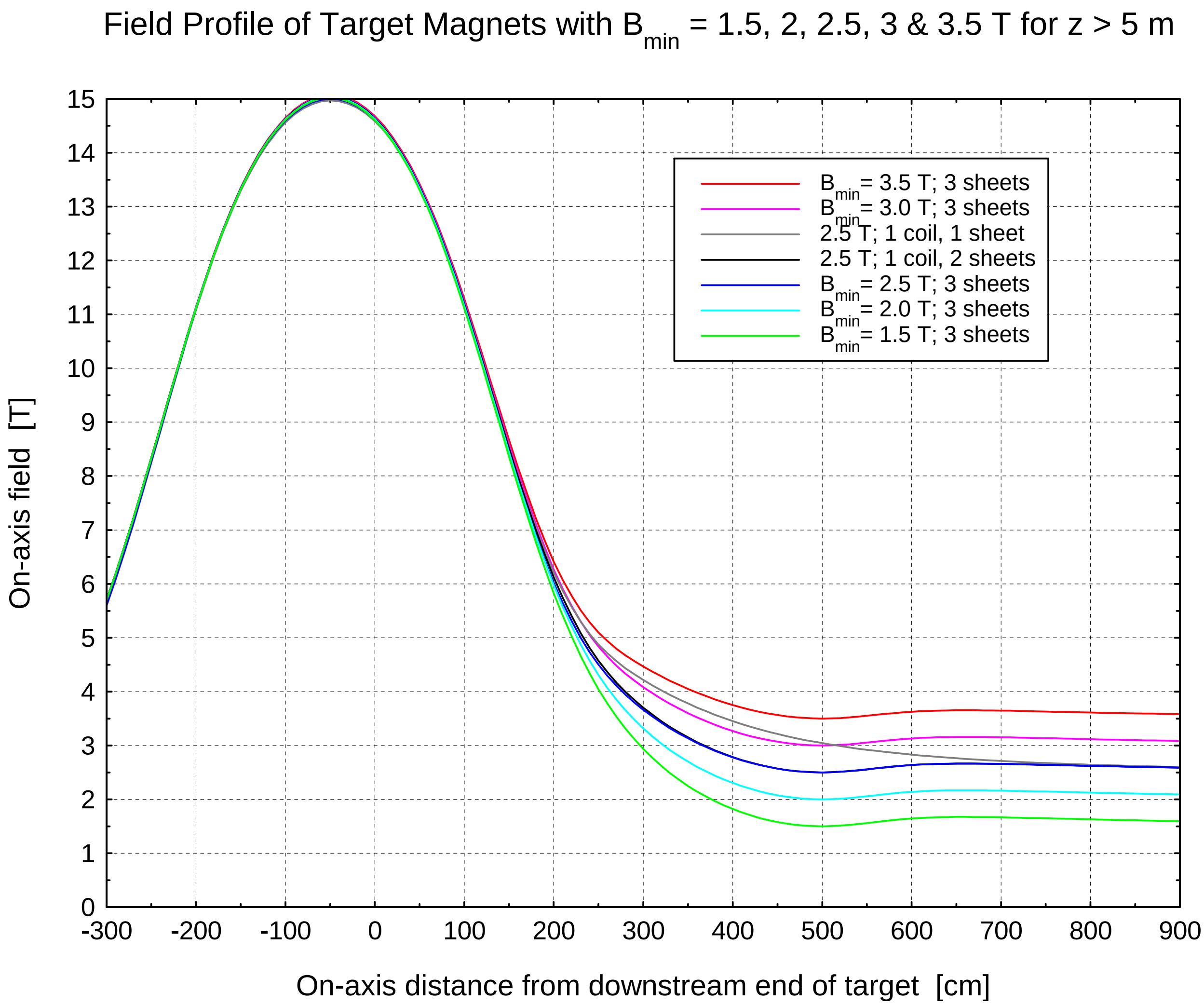

Fig. 1. On-axis field profiles generated by equations described in text above.

The field near z = 6 m is higher than desired by as much as 0.17 T (0.5 T for the design with only one downstream coil). The field profiles of Figure 7 reduce this field deviation by means of an additional coil and therefore a second adjustable current. The solution of two simultaneous linear equations suffices to determine the currents in the interior two coils that satisfy the constraints $B(z) = B_{min}$ at z = 5 m and at z = 6 m. The latter constraint is more robust than requiring dB/dz = 0 at z = 5 m.

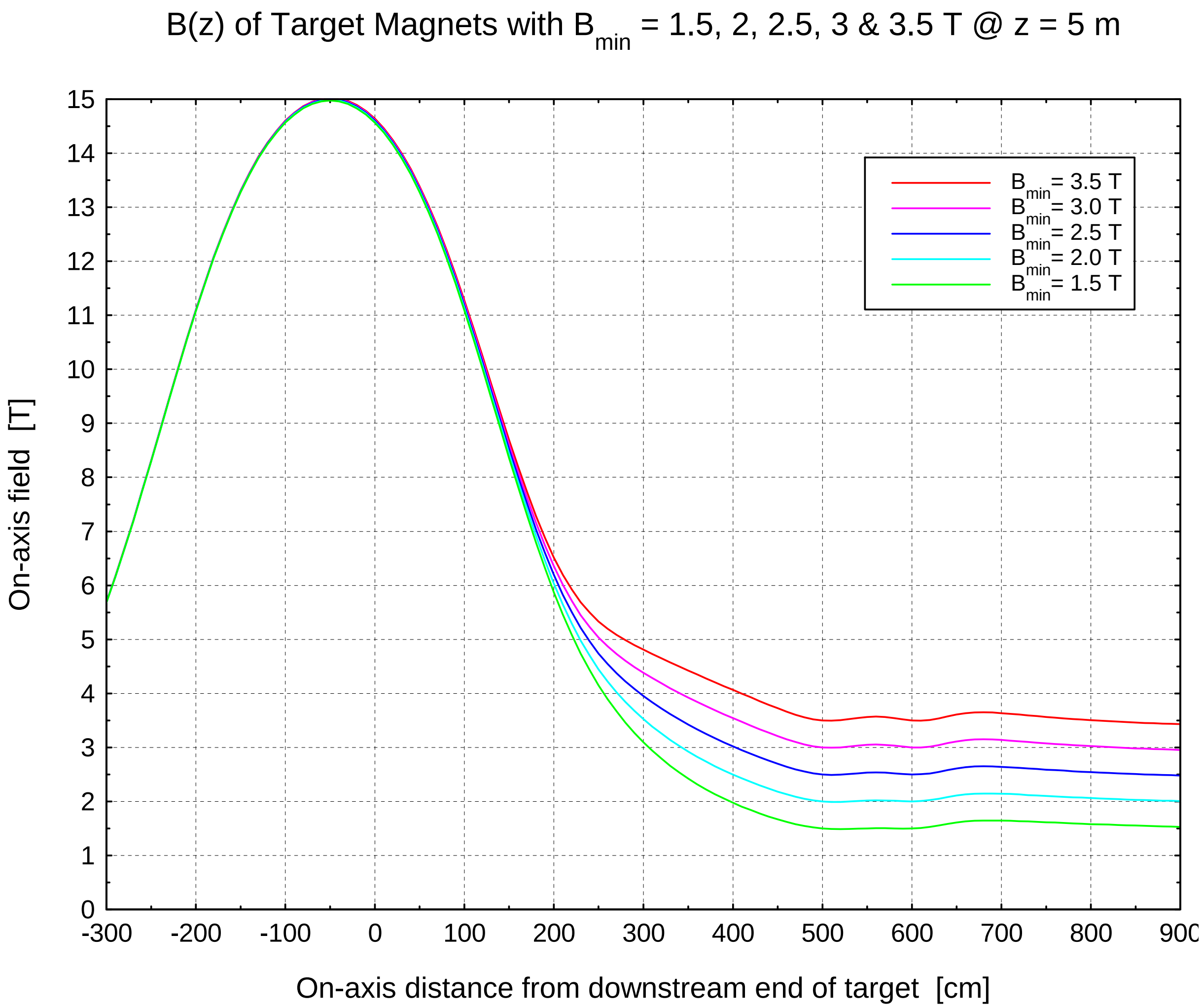

Fig. 2. On-axis field profiles generated by magnets with four current sheets. Currents i_1 and i_4 are fixed; i_2 and i_3 are adjustable. $B(z) = B_{min}$ at $z = 5$ m; also $B(6m) = B_{min}$, a more robust constraint than $dB/dz = 0$ at $z = 5m$.

All the designs employ four current sheets with consecutive radii of [1.64, 1.00, 0.75, 0.60] m; coil ends are at [−2.35, 2.50, 5.29, 6.28] m upstream and [1.34, 5.17, 6.18, ∞] m downstream. To make the total field 15 T at z = −0.5m), the upstream coil generates 14.97 T. To make the total field approximately B_{min} throughout the range 5 m to ~9 m, the downstream coil carries 94% the current that it would need if unassisted by the coils upstream.

Design Number	#1	#2	#3	#4	#5
Desired field @ z = L = 5 m [T]	1.5	2.0	2.5	3.0	3.5
B_∞ of current sheet #2 [T]	0.918	1.536	2.153	2.770	3.387
B_∞ of current sheet #3 [T]	1.226	1.802	2.377	2.953	3.528

The on-axis field B at location z generated by a current sheet of radius a, upstream end b_1, downstream end b_2, and carrying i' amperes per meter is the function

$$B(z) = \tfrac{1}{2}\mu_0\, i'\, (b-z)\, [a^2+(b-z^2)]^{-1/2}\ \text{teslas}$$

evaluated at $b = b_2$ minus its evaluation at $b = b_1$.

Define a function $B^{(0)}(x) \equiv x(a^2+x^2)^{-1/2}$, where $x \equiv (b-z)$. Similarly, define functions $B^{(n)}(x) \equiv (1/n!)\, d^nB^{(0)}/dx^n$. Then the Taylor series expansion of the field an increment Δ from z is the function

$$\tfrac{1}{2}\mu_0\, i'\, [B^{(0)}(x) + B^{(1)}(x)\,\Delta + B^{(2)}(x)\,\Delta^2 + B^{(3)}(x)\,\Delta^3 + B^{(4)}(x)\,\Delta^4 + B^{(5)}(x)\,\Delta^5 + \ldots],$$

evaluated at $x_1 \equiv (b_1-z)$ and subtracted from its evaluation at $x_2 \equiv (b_2-z)$.

Define $r = (a^2+x^2)^{1/2}$. The first five terms in the series are:

$$B^{(1)}(x) = a^2 / r^3\,,$$

$$B^{(2)}(x) = -(3/2)\, a^2\, x / r^5\,,$$

$$B^{(3)}(x) = -(1/2)\, (a^2 - 4\,x^2) / r^7\,,$$

$$B^{(4)}(x) = (5/8)\, a^2\, x\, (3\,a^2 - 4\,x^2) / r^9\,,$$

$$B^{(5)}(x) = (3/8)\, a^2\, (a^4 - 12\,a^2\,x^2 + 8\,x^4) / r^{11}\,.$$

Figure 3 plots these functions with $a = 1$. To devote most of the graph to values of x less than unity, values greater than unity are transformed as $x = (2-u)^{-1}$, where u is the abscissa. The right-hand border of the graph, $u = 1.8$, transforms to $x = 5$.

The figure also plots the power-series coefficients for the function $B^{(0)}(x) = \tanh(x)$, which behaves much like $(1+x^2)^{-1/2}$. Its coefficients are:

$$B^{(1)}(x) = 1 / \cosh^2(x)\,,$$

$$B^{(2)}(x) = -\sinh(x) / \cosh^3(x)\,,$$

$$B^{(3)}(x) = -(1/3)\, [1 - 2\sinh^2(x)] / \cosh^4(x)\,,$$

$$B^{(4)}(x) = (1/3)\, [\sinh(x)]\, [2 - \sinh^2(x)] / \cosh^5(x)\,,$$

$$B^{(5)}(x) = (2/15)\, [2 - 11\sinh^2(x) + 2\sinh^4(x)] / \cosh^6(x)\,.$$

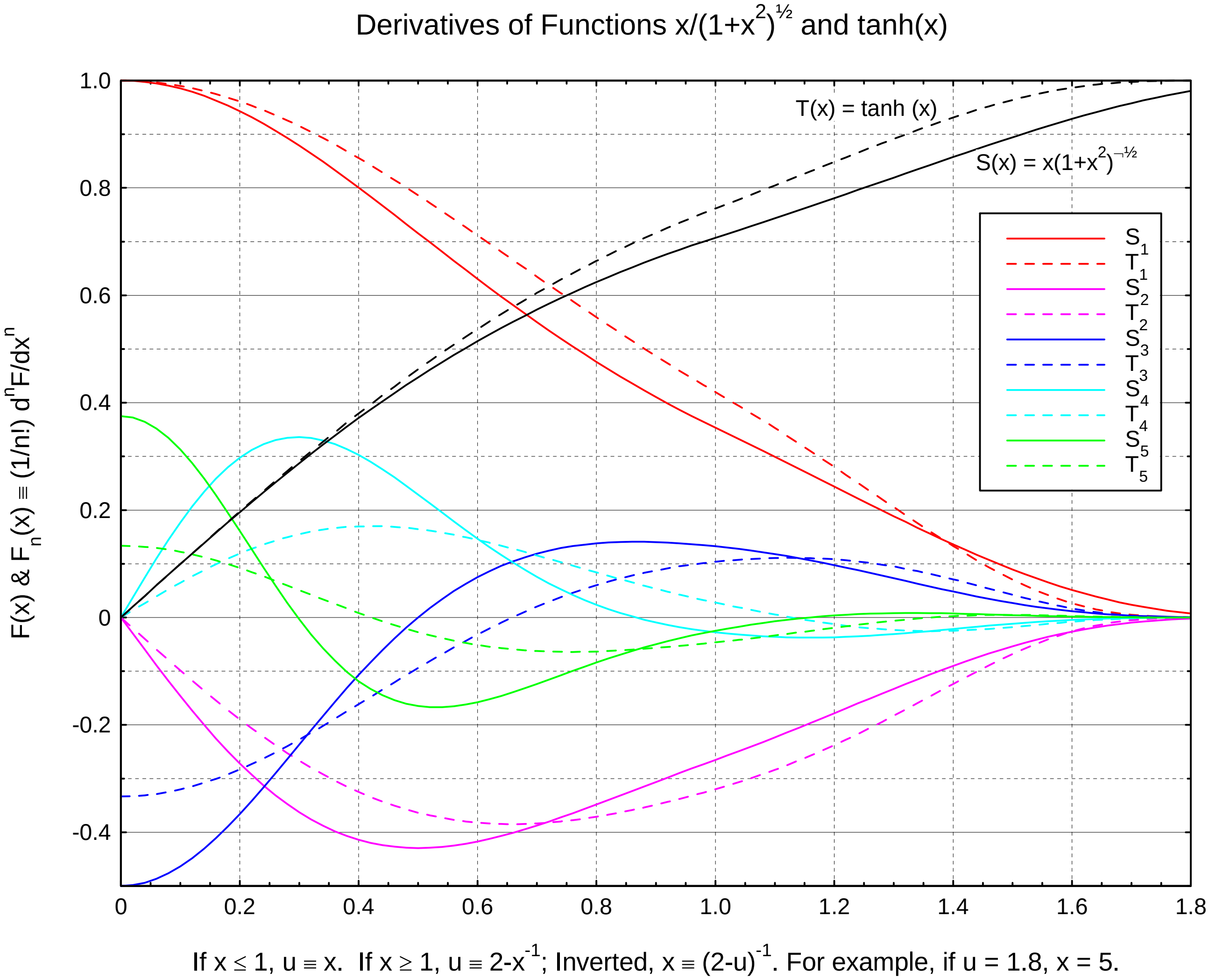

Fig. 3. Field and power-series coefficients of current sheet and function tanh(x), which behaves much like $(1+x^2)^{-½}$.