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Bunch-Timing Measurement in the Muon Cooling Experiment Via Rectangular $\mathrm{TE}_{0,1,n}$ RF Cavities

1 Introduction

Rick Fernow has suggested the use of an RF cavity to impart a transverse displacement proportional to the longitudinal offset of a muon from the center of the bunch. This would permit inference of the z and t coordinates of a muon via measurement of transverse coordinates. Here we sketch the performance of such a device based on analytic approximations, which have been confirmed for particular cases by numerical integration of the muon trajectories through the RF cavity.

We find that a good choice might be an 800-MHz rectangular $\mathrm{TE}_{0,1,1}$ cavity of length 20 cm along the beam and 54 cm transverse to the beam, for which the transverse displacement is 6.2 μm/ps, assuming a peak field strength of 40 MV/m. However, multiple scattering in the cavity walls, as well as in the tracking system, may limit the timing resolution to many tens of picoseconds.

2 Generalities

The rf cavity is centered on $(x, y, z) = (0, 0, 0)$. To begin we suppose the cavity is a rectangular box of length b in x (the direction of transverse deflection) and length a in y and z. In secs. 3.3-6 we generalize the results to cavities of arbitrary aspect ratio in y and z.

The trajectory of a typical beam particle for the cavity field OFF is parametrized as

$$\begin{aligned} x &= x_0 + \beta_x ct, \\ y &= y_0 + \beta_y ct, \\ z &= z_0 + \beta_z ct, \end{aligned} \tag{1}$$

where c is the speed of light. The beam axis is the z-axis:

$$\beta_x,\ \beta_y \ll \beta_z, \qquad \text{and} \qquad \beta_z \approx \beta. \tag{2}$$

We will make the impulse approximation that the cavity fields do not affect the muon trajectories in y or z, but only in x. Thus we assume the y and z parametrizations in (1) also hold when the field in ON.

The particle is within the cavity during the interval

$$[t_{\rm min}, t_{\rm max}] = \left[-\frac{a}{2\beta_z c} - \frac{z_0}{\beta_z c}, \frac{a}{2\beta_z c} - \frac{z_0}{\beta_z c}\right] = \left[-\frac{a}{2\beta_z c}\left(1 + \frac{2z_0}{a}\right), \frac{a}{2\beta_z c}\left(1 - \frac{2z_0}{a}\right)\right], \qquad (3)$$

so that

$$t_{\rm max} - t_{\rm min} = \frac{a}{\beta_z c}. \qquad (4)$$

2.1 Square $\mathrm{TE}_{0,1,1}$ Cavity Fields

As our first example we consider a $\mathrm{TE}_{0,1,1}$ RF cavity operating at angular frequency ω and phased such that the electric field is maximum at $t = 0$, *i.e.*, when the center of the bunch is at the center of the cavity. The wave equation tells us that

$$\frac{\omega}{c} = \sqrt{2}\frac{\pi}{a}, \qquad \text{so} \qquad a = \frac{\sqrt{2}c}{2f} = 26.5 \text{ cm for } f = 800 \text{ MHz}. \qquad (5)$$

In Gaussian units the fields are

$$\begin{aligned} E_x &= E_0 \cos\frac{\pi y}{a}\cos\frac{\pi z}{a}\cos\omega t, \\ E_y = E_z &= 0, \\ B_x &= 0, \\ B_y &= +\frac{E_0}{\sqrt{2}}\cos\frac{\pi y}{a}\sin\frac{\pi z}{a}\sin\omega t, \\ B_z &= -\frac{E_0}{\sqrt{2}}\sin\frac{\pi y}{a}\cos\frac{\pi z}{a}\sin\omega t, \end{aligned} \qquad (6)$$

where E_0 is the peak electric field.

2.2 Square $\mathrm{TE}_{0,1,2}$ Cavity Fields

The wave equation tells us that for such a cavity

$$\frac{\omega}{c} = \sqrt{5}\frac{\pi}{a}, \qquad \text{so} \qquad a = \frac{\sqrt{5}c}{2f} = 42.0 \text{ cm for } f = 800 \text{ MHz}. \qquad (7)$$

The cavity is phased so that the electric field is minimum at $t = 0$:

$$\begin{aligned} E_x &= E_0 \cos\frac{\pi y}{a}\sin\frac{2\pi z}{a}\sin\omega t, \\ E_y = E_z &= 0, \\ B_x &= 0, \\ B_y &= \frac{2E_0}{\sqrt{5}}\cos\frac{\pi y}{a}\cos\frac{2\pi z}{a}\cos\omega t, \\ B_z &= \frac{E_0}{\sqrt{5}}\sin\frac{\pi y}{a}\sin\frac{2\pi z}{a}\cos\omega t. \end{aligned} \qquad (8)$$

3 Transverse Displacement: Leading Approximation

To a good approximation the energy of the muon does not change in the RF cavity; $\gamma = 1/\sqrt{1-\beta^2}$ remains constant. The x-component of the Lorentz-force law can then be written

$$\frac{dv_x}{dt} = \frac{e}{\gamma m}(E_x + \beta_y B_z - \beta_z B_y). \tag{9}$$

The change in the x-velocity due to the RF cavity is

$$\Delta v_x = \int_{t_{\rm min}}^{t} \frac{dv_x}{dt'} dt', \tag{10}$$

and the desired transverse displacement across the cavity is

$$\Delta x = \int_{t_{\rm min}}^{t_{\rm max}} dt' \int_{t_{\rm min}}^{t'} \frac{dv_x}{dt''} dt''. \tag{11}$$

3.1 Square TE$_{0,1,1}$ Cavity

Using the fields (6) in the Lorentz force we have

$$\begin{aligned} \frac{dv_x}{dt} = \ & \frac{eE_0}{\gamma m}\Big\{\cos\frac{\pi(y_0+\beta_y ct)}{a}\cos\frac{\pi(z_0+\beta_z ct)}{a}\cos\omega t - \frac{\beta_y}{\sqrt{2}}\sin\frac{\pi(y_0+\beta_y ct)}{a}\cos\frac{\pi(z_0+\beta_z ct)}{a}\sin\omega t \\ & -\frac{\beta_z}{\sqrt{2}}\cos\frac{\pi(y_0+\beta_y ct)}{a}\sin\frac{\pi(z_0+\beta_z ct)}{a}\sin\omega t\Big\}. \end{aligned} \tag{12}$$

This expression is independent of x and β_x (except for the quadratic dependence of β_z on β_x when β is constant). We will suppose that

$$\frac{\pi(y_0 + \beta_y ct)}{a} \ll 1. \tag{13}$$

Then the dependence of the x-displacement on y and β_y is through the second-order quantities β_y^2 and $y_0\beta_y/a$. However, the argument of the functions of z is not small: from (4) and (5) we see that $\omega(t_{\rm max} - t_{\rm min}) = \sqrt{2}\pi/\beta_z > \pi$.

Then to the first approximation,

$$\frac{dv_x}{dt} \approx \frac{eE_0}{\gamma m}\left\{\cos\frac{\pi(z_0+\beta_z ct)}{a}\cos\omega t + \frac{\beta_z}{\sqrt{2}}\sin\frac{\pi(z_0+\beta_z ct)}{a}\sin\omega t\right\}. \tag{14}$$

It is useful to introduce the notation

$$\omega' = \frac{\pi\beta_z c}{a} = \frac{\beta_z\omega}{\sqrt{2}}, \qquad \text{and} \qquad \epsilon = \frac{\pi z_0}{a}, \tag{15}$$

so that

$$\frac{\pi(z_0+\beta_z ct)}{a} = \omega' t + \epsilon, \qquad \text{and} \qquad \omega \pm \omega' = \frac{\sqrt{2}\pm\beta_z}{\sqrt{2}}\omega. \tag{16}$$

We now assume that $\epsilon \ll 1$, *i.e.*, that the bunch length is small compared to the cavity length. Then

$$\begin{aligned}\frac{dv_x}{dt} &\approx \frac{eE_0}{\gamma m}\left\{\cos(\omega' t+\epsilon)\cos\omega t + \frac{\beta_z}{\sqrt{2}}\sin(\omega' t+\epsilon)\sin\omega t\right\} \\ &= \frac{eE_0}{2\gamma m}\left\{\frac{\sqrt{2}+\beta_z}{\sqrt{2}}\cos[(\omega+\omega')t+\epsilon] + \frac{\sqrt{2}-\beta_z}{\sqrt{2}}\cos[(\omega-\omega')t-\epsilon]\right\} \qquad (17)\end{aligned}$$

We integrate this from $t_{\min}$ to t to find the velocity kick,

$$\begin{aligned}\Delta v_x &\approx \frac{eE_0}{2\gamma m\omega}\Big\{\sin[(\omega+\omega')t+\epsilon] + \sin[(\omega-\omega')t-\epsilon] \\ &\qquad - \sin[(\omega+\omega')t_{\min}+\epsilon] - \sin[(\omega-\omega')t_{\min}-\epsilon]\Big\} \qquad (18)\end{aligned}$$

We integrate (18) from $t_{\min}$ to $t_{\max}$ to find the x-displacement due to the RF fields:

$$\begin{aligned}\Delta x &\approx \frac{eE_0}{2\gamma m w}\Big\{\frac{\sqrt{2}}{(\sqrt{2}+\beta_z)\omega}\left(-\cos[(\omega+\omega')t_{\max}+\epsilon] + \cos[(\omega+\omega')t_{\min}+\epsilon]\right) \\ &\quad + \frac{\sqrt{2}}{(\sqrt{2}-\beta_z)\omega}\left(-\cos[(\omega-\omega')t_{\max}-\epsilon] + \cos[(\omega-\omega')t_{\min}-\epsilon]\right) \\ &\quad - \left(\sin[(\omega+\omega')t_{\min}+\epsilon] + \sin[(\omega-\omega')t_{\min}-\epsilon]\right)(t_{\max}-t_{\min})\Big\}. \qquad (19)\end{aligned}$$

To evaluate the various terms in (19) we note from (3), (15) and (16) that

$$(\omega+\omega')t_{\max}+\epsilon = \frac{\pi}{2}\left(\frac{\sqrt{2}}{\beta_z}+1\right) - \frac{\sqrt{2}\epsilon}{\beta_z}. \qquad (20)$$

Hence

$$\begin{aligned}\cos[(\omega+\omega')t_{\max}+\epsilon] &\approx \cos\frac{\pi}{2}\left(\frac{\sqrt{2}}{\beta_z}+1\right) + \frac{\sqrt{2}\epsilon}{\beta_z}\sin\frac{\pi}{2}\left(\frac{\sqrt{2}}{\beta_z}+1\right) \\ &= -\sin\frac{\sqrt{2}\pi}{2\beta_z} + \frac{\sqrt{2}\epsilon}{\beta_z}\cos\frac{\sqrt{2}\pi}{2\beta_z}. \qquad (21)\end{aligned}$$

Similarly,

$$\sin[(\omega+\omega')t_{\max}+\epsilon] \approx \cos\frac{\sqrt{2}\pi}{2\beta_z} + \frac{\sqrt{2}\epsilon}{\beta_z}\sin\frac{\sqrt{2}\pi}{2\beta_z}, \qquad (22)$$

$$(\omega+\omega')t_{\min}+\epsilon = -\frac{\pi}{2}\left(\frac{\sqrt{2}}{\beta_z}+1\right) - \frac{\sqrt{2}\epsilon}{\beta_z}, \qquad (23)$$

$$\cos[(\omega+\omega')t_{\min}+\epsilon] \approx -\sin\frac{\sqrt{2}\pi}{2\beta_z} - \frac{\sqrt{2}\epsilon}{\beta_z}\cos\frac{\sqrt{2}\pi}{2\beta_z}, \qquad (24)$$

$$\sin[(\omega+\omega')t_{\min}+\epsilon] \approx -\cos\frac{\sqrt{2}\pi}{2\beta_z} + \frac{\sqrt{2}\epsilon}{\beta_z}\sin\frac{\sqrt{2}\pi}{2\beta_z}, \qquad (25)$$

$$(\omega - \omega')t_{\max} - \epsilon = \frac{\pi}{2}\left(\frac{\sqrt{2}}{\beta_z} - 1\right) - \frac{\sqrt{2}\epsilon}{\beta_z}, \tag{26}$$

$$\cos[(\omega - \omega')t_{\max} - \epsilon] \approx \sin\frac{\sqrt{2}\pi}{2\beta_z} - \frac{\sqrt{2}\epsilon}{\beta_z}\cos\frac{\sqrt{2}\pi}{2\beta_z}, \tag{27}$$

$$\sin[(\omega - \omega')t_{\max} - \epsilon] \approx -\cos\frac{\sqrt{2}\pi}{2\beta_z} - \frac{\sqrt{2}\epsilon}{\beta_z}\sin\frac{\sqrt{2}\pi}{2\beta_z}, \tag{28}$$

$$(\omega - \omega')t_{\min} - \epsilon = -\frac{\pi}{2}\left(\frac{\sqrt{2}}{\beta_z} - 1\right) - \frac{\sqrt{2}\epsilon}{\beta_z}, \tag{29}$$

$$\cos[(\omega - \omega')t_{\min} - \epsilon] \approx \sin\frac{\sqrt{2}\pi}{2\beta_z} + \frac{\sqrt{2}\epsilon}{\beta_z}\cos\frac{\sqrt{2}\pi}{2\beta_z}, \tag{30}$$

and

$$\sin[(\omega - \omega')t_{\min} - \epsilon] \approx \cos\frac{\sqrt{2}\pi}{2\beta_z} - \frac{\sqrt{2}\epsilon}{\beta_z}\sin\frac{\sqrt{2}\pi}{2\beta_z}. \tag{31}$$

Using (21-31) in (19) and recalling (5) and (15) we find the main result,

$$\Delta x \approx \frac{2\sqrt{2}}{\gamma(2 - \beta_z^2)}\eta z_0 \cos\frac{\sqrt{2}\pi}{2\beta_z}, \tag{32}$$

in which we have introduced the dimensionless measure of field strength

$$\eta = \frac{eE_0}{m\omega c} = \frac{eE_0}{mc^2}\frac{c}{2\pi f}, \tag{33}$$

with f as the cavity frequency.

We can also evaluate the transverse-velocity kick at time $t_{\max}$ using eqs. (18) and (21-31), to find that it vanishes to the first approximation. The muons enter and exit the cavity with the same x-slope.

We can now consider a numerical example based on the proposed muon cooling experiment. The nominal muon momentum is 165 MeV/c, for which $\gamma = 1.85$ and $\beta = 0.84$. In this case $\cos(\sqrt{2}\pi/2\beta_z) = -0.87$.

The nominal frequency of the RF cavity is $f = 800$ MHz. I suppose that we can achieve a field strength of 40 MV/m in the cavity. For a muon $mc^2 = 105.7$ MeV, so

$$\eta = \frac{40}{105.7}\frac{3}{16\pi} = 0.0223. \tag{34}$$

Combining these factors, eq. (33) becomes

$$\Delta x = -\frac{2\sqrt{2} \cdot 0.0223 \cdot 0.87}{1.85 \cdot 1.29} z_0 = -0.023 z_0. \tag{35}$$

We can convert this to a relation involving the time offset,

$$\Delta t = \frac{z_0}{\beta_z c}, \tag{36}$$

to find

$$\Delta x = -0.023\beta_z c\Delta t = -0.019 c\Delta t = -5.8[\mu\text{m}]\left[\frac{\Delta t}{1\ \text{ps}}\right]. \tag{37}$$

3.2 Square $\mathrm{TE}_{0,1,2}$ Cavity

Using the fields (8) in the Lorentz force we have

$$\frac{dv_x}{dt} = \frac{eE_0}{\gamma m}\left\{\cos\frac{\pi(y_0+\beta_y ct)}{a}\sin\frac{\pi(z_0+\beta_z ct)}{a}\sin\omega t + \frac{\beta_y}{\sqrt{5}}\sin\frac{\pi(y_0+\beta_y ct)}{a}\sin\frac{2\pi(z_0+\beta_z ct)}{a}\cos\omega t \right.$$
$$\left. -\frac{2\beta_z}{\sqrt{5}}\cos\frac{\pi(y_0+\beta_y ct)}{a}\cos\frac{\pi(z_0+\beta_z ct)}{a}\cos\omega t\right\}. \quad (38)$$

To the first approximation:

$$\frac{dv_x}{dt} \approx \frac{eE_0}{\gamma m}\left\{\sin\frac{2\pi(z_0+\beta_z ct)}{a}\sin\omega t - \frac{2\beta_z}{\sqrt{5}}\cos\frac{2\pi(z_0+\beta_z ct)}{a}\cos\omega t\right\}$$
$$= \frac{eE_0}{2\gamma m}\left\{-\frac{\sqrt{5}+2\beta_z}{\sqrt{5}}\cos[(\omega+\omega')t+\epsilon] + \frac{\sqrt{5}-2\beta_z}{\sqrt{5}}\cos[(\omega-\omega')t-\epsilon]\right\}, \quad (39)$$

where we have introduced the notation

$$\omega' = \frac{2\pi\beta_x c}{a} = \frac{2\beta_z\omega}{\sqrt{5}}, \qquad \text{and} \qquad \epsilon = \frac{2\pi z_0}{a}, \quad (40)$$

so that

$$\frac{2\pi(z_0+\beta_z ct)}{a} = \omega' t+\epsilon, \qquad \text{and} \qquad \omega\pm\omega' = \frac{\sqrt{5}\pm 2\beta_z}{\sqrt{5}}\omega. \quad (41)$$

We integrate (39) from t_{min} to t to find the velocity kick,

$$\Delta v_x \approx \frac{eE_0}{2\gamma m\omega}\Big\{-\sin[(\omega+\omega')t+\epsilon] + \sin[(\omega-\omega')t-\epsilon]$$
$$+\sin[(\omega+\omega')t_{\mathrm{min}}+\epsilon] - \sin[(\omega-\omega')t_{\mathrm{min}}-\epsilon]\Big\} \quad (42)$$

We integrate (42) from t_{min} to t_{max} to find the x-displacement due to the RF fields:

$$\Delta x \approx \frac{eE_0}{2\gamma m w}\left\{\frac{\sqrt{5}}{(\sqrt{5}+2\beta_z)\omega}\left(\cos[(\omega+\omega')t_{\mathrm{max}}+\epsilon] - \cos[(\omega+\omega')t_{\mathrm{min}}+\epsilon]\right)\right.$$
$$-\frac{\sqrt{5}}{(\sqrt{5}-2\beta_z)\omega}\left(\cos[(\omega-\omega')t_{\mathrm{max}}-\epsilon] - \cos[(\omega-\omega')t_{\mathrm{min}}-\epsilon]\right)$$
$$\left.+\left(\sin[(\omega+\omega')t_{\mathrm{min}}+\epsilon] - \sin[(\omega-\omega')t_{\mathrm{min}}-\epsilon]\right)(t_{\mathrm{max}}-t_{\mathrm{min}})\right\}. \quad (43)$$

To evaluate the various terms in (43) we note from (3), (40) and (41) that

$$(\omega+\omega')t_{\mathrm{max}}+\epsilon = \frac{\pi}{2}\left(\frac{\sqrt{5}}{\beta_z}+2\right) - \frac{\sqrt{5}\epsilon}{2\beta_z}, \quad (44)$$

$$\cos[(\omega+\omega')t_{\mathrm{max}}+\epsilon] \approx -\cos\frac{\sqrt{5}\pi}{2\beta_z} - \frac{\sqrt{5}\epsilon}{2\beta_z}\sin\frac{\sqrt{5}\pi}{2\beta_z}, \quad (45)$$

$$(\omega + \omega')t_{\min} + \epsilon = -\frac{\pi}{2}\left(\frac{\sqrt{5}}{\beta_z} + 2\right) - \frac{\sqrt{5}\epsilon}{2\beta_z}, \tag{46}$$

$$\cos[(\omega + \omega')t_{\min} + \epsilon] \approx -\cos\frac{\sqrt{5}\pi}{2\beta_z} + \frac{\sqrt{5}\epsilon}{2\beta_z}\sin\frac{\sqrt{5}\pi}{2\beta_z}, \tag{47}$$

$$\sin[(\omega + \omega')t_{\min} + \epsilon] \approx \sin\frac{\sqrt{5}\pi}{2\beta_z} + \frac{\sqrt{5}\epsilon}{2\beta_z}\cos\frac{\sqrt{5}\pi}{2\beta_z}, \tag{48}$$

$$(\omega - \omega')t_{\max} - \epsilon = \frac{\pi}{2}\left(\frac{\sqrt{5}}{\beta_z} - 2\right) - \frac{\sqrt{5}\epsilon}{2\beta_z}, \tag{49}$$

$$\cos[(\omega - \omega')t_{\max} - \epsilon] \approx -\cos\frac{\sqrt{5}\pi}{2\beta_z} - \frac{\sqrt{5}\epsilon}{2\beta_z}\sin\frac{\sqrt{5}\pi}{2\beta_z}, \tag{50}$$

$$(\omega - \omega')t_{\min} - \epsilon = -\frac{\pi}{2}\left(\frac{\sqrt{5}}{\beta_z} - 2\right) - \frac{\sqrt{5}\epsilon}{2\beta_z}, \tag{51}$$

$$\cos[(\omega - \omega')t_{\min} - \epsilon] \approx -\cos\frac{\sqrt{5}\pi}{2\beta_z} + \frac{\sqrt{5}\epsilon}{2\beta_z}\sin\frac{\sqrt{5}\pi}{2\beta_z}, \tag{52}$$

and

$$\sin[(\omega - \omega')t_{\min} - \epsilon] \approx \sin\frac{\sqrt{5}\pi}{2\beta_z} + \frac{\sqrt{5}\epsilon}{2\beta_z}\cos\frac{\sqrt{5}\pi}{2\beta_z}. \tag{53}$$

Using (45-53) in (19) and recalling (7) and (41) we find

$$\Delta x \approx \frac{4\sqrt{5}}{\gamma(5 - 4\beta_z^2)}\eta z_0 \sin\frac{\sqrt{5}\pi}{2\beta_z}. \tag{54}$$

For $\beta = 0.84$ we have $\sin(\sqrt{5}\pi/2\beta_z) = -0.86$. Thus for $E_0 = 40$ MV/m

$$\Delta x = -\frac{4\sqrt{5}\cdot 0.0223\cdot 0.86}{1.85\cdot 2.18}z_0 = -0.0425 z_0 = -0.0357 c\Delta t = -10.7[\mu\text{m}]\left[\frac{\Delta t}{1\text{ ps}}\right]. \tag{55}$$

3.3 General $\text{TE}_{0,1,1}$ Cavity

The cavity length in z is still called a, but the length in y is a/α where α will be chosen to optimize Δx for a given β_z.

The wave equation tells us

$$\frac{\omega}{c} = \sqrt{1 + \alpha^2}\frac{\pi}{a}. \tag{56}$$

The results of sec. 3.1 hold on substituting $\sqrt{1 + \alpha^2}$ for $\sqrt{2}$:

$$\frac{\Delta x}{\eta z_0} \approx \frac{2\sqrt{1 + \alpha^2}}{\gamma(1 + \alpha^2 - \beta_z^2)}\cos\frac{\sqrt{1 + \alpha^2}\pi}{2\beta_z} = \frac{2\gamma\sqrt{1 + \alpha^2}}{1 + \gamma^2\alpha^2}\cos\frac{\sqrt{1 + \alpha^2}\pi}{2\beta_z}. \tag{57}$$

Expression (57) has a broad maximum at $\alpha = 0.58$, as shown in Fig. 1. For further discussion, see sec. 3.6.

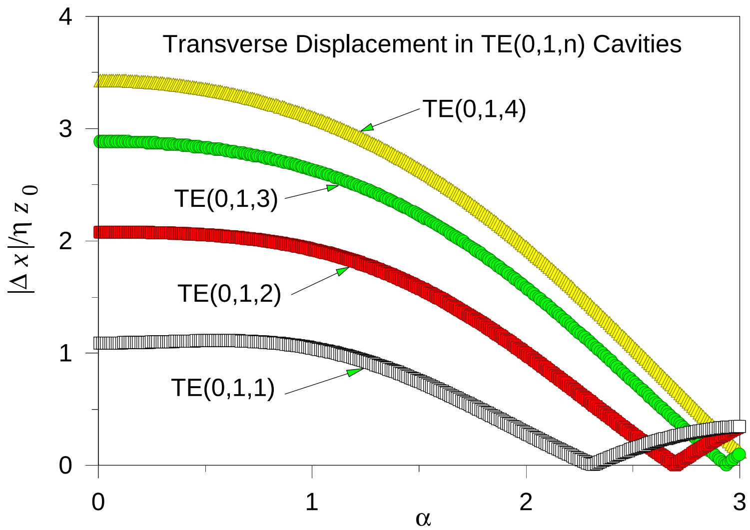

Figure 1: Transverse displacement in 800-MHz rectangular $\mathrm{TE}_{0,1,n}$ RF cavities as a function of the aspect ratio $\alpha = z$-length/y-length, for $\beta_z = 0.84$.

3.4 General $\mathrm{TE}_{0,1,2}$ Cavity

The cavity length in z is still called a, but the length in y is a/α where α will be chosen to maximize Δx for a given β_z.

The wave equation tells us

$$\frac{\omega}{c} = \sqrt{4+\alpha^2}\frac{\pi}{a}. \tag{58}$$

The results of sec. 3.2 hold on substituting $\sqrt{4+\alpha^2}$ for $\sqrt{5}$:

$$\frac{\Delta x}{\eta z_0} \approx \frac{4\sqrt{4+\alpha^2}}{\gamma(4+\alpha^2-4\beta_z^2)} \sin\frac{\sqrt{4+\alpha^2}\pi}{2\beta_z} = \frac{2\gamma\sqrt{1+(\alpha/2)^2}}{1+(\gamma\alpha/2)^2} \sin\frac{\sqrt{2^2+(\alpha/2)^2}\pi}{2\beta_z}. \tag{59}$$

As shown in Fig. 1, expression (59) is nearly constant for $\alpha \le 1$, and falls off for $\alpha > 1$; $\Delta x = 0$ for $\alpha = 2.7$. Thus $\alpha = 1$ is a reasonable choice for a rectangular $\mathrm{TE}_{0,1,2}$ cavity, in which case $\Delta x/\eta z_0 = -1.92$.

3.5 General $\mathrm{TE}_{0,1,n}$ Cavity

The wave equation tells us

$$\frac{\omega}{c} = \sqrt{n^2+\alpha^2}\frac{\pi}{a}. \tag{60}$$

Without having gone through the details, I conjecture from eqs. (57) and (59) that the result is

$$\frac{\Delta x}{\eta z_0} \approx \frac{2\gamma\sqrt{1+(\alpha/n)^2}}{1+(\gamma\alpha/n)^2} f\left(\frac{\sqrt{n^2+\alpha^2}\pi}{2\beta_z}\right), \qquad \text{where } f = \begin{cases} \cos, & n \text{ odd}, \\ \sin, & n \text{ even}. \end{cases} \tag{61}$$

3.6 Discussion

The transverse displacement Δx must be measured between the entrance and exit of the RF cavity, *i.e.*, over the length a along the z-axis. Multiple scattering will limit the accuracy of this measurement, so the ratio $|\Delta x|/a\eta z_0$ is the relevant figure of merit.

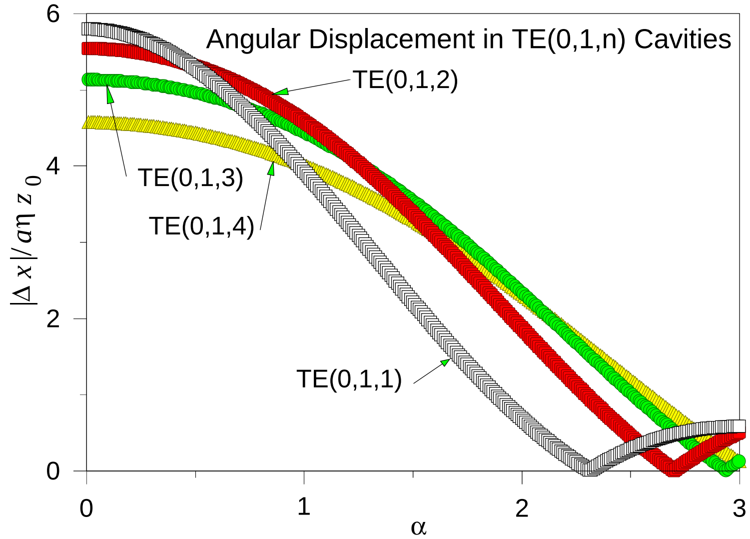

Figure 2: Transverse displacement divided by the cavity length in 800-MHz rectangular $\mathrm{TE}_{0,1,n}$ RF cavities as a function of the aspect ratio α, for $\beta_z = 0.84$.

Figures 2-4 show the ratio $|\Delta x|/a\eta z_0$ as a function of α, a and a/α, respectively. We see that a while square $\mathrm{TE}_{0,1,1}$ is less effective than square $\mathrm{TE}_{0,1,2}$ and $\mathrm{TE}_{0,1,3}$ cavities, a $\mathrm{TE}_{0,1,1}$ cavity is superior once $\alpha < 0.5$.

For very small α the cavity height becomes very large, so it would not be practical to approach this limit. Figure 5 shows the relation between cavity height and length.

An interesting option might be a rectangular $\mathrm{TE}_{0,1,1}$ cavity with $\alpha = 0.37$, corresponding to length $a = 20$ cm, height $a/\alpha = 54$ cm, transverse displacement $\Delta x/\eta z_0 = -1.10$ and angular displacement $\Delta x/a\eta z_0 = -5.50$. However, for a peak field of 40 MV/m, $\eta = 0.0223$ and the transverse displacement is only

$$|\Delta x| = 6.2[\mu\mathrm{m}]\left[\frac{\Delta t}{1\ \mathrm{ps}}\right], \tag{62}$$

corresponding to angular displacement

$$\frac{|\Delta x|}{a} = 30[\mu\mathrm{rad}]\left[\frac{\Delta t}{1\ \mathrm{ps}}\right]. \tag{63}$$

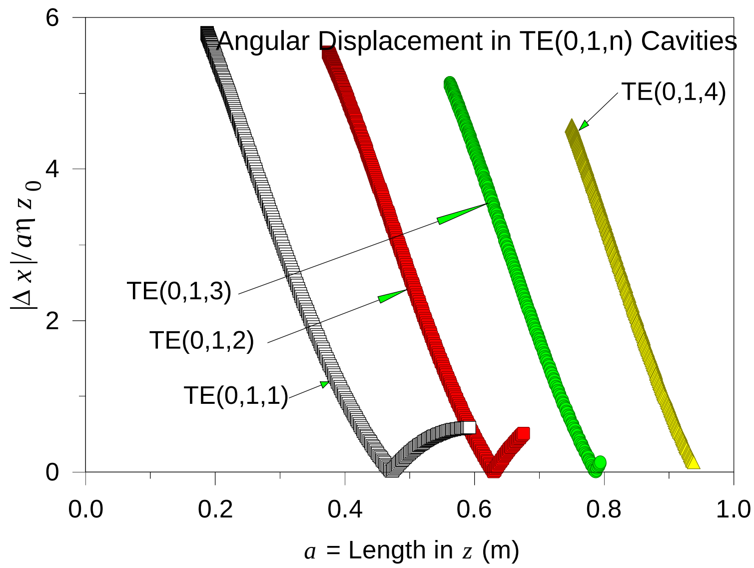

Figure 3: Transverse displacement divided by the cavity length in 800-MHz rectangular $TE_{0,1,n}$ RF cavities as a function of the cavity length a, for $\beta_z = 0.84$.

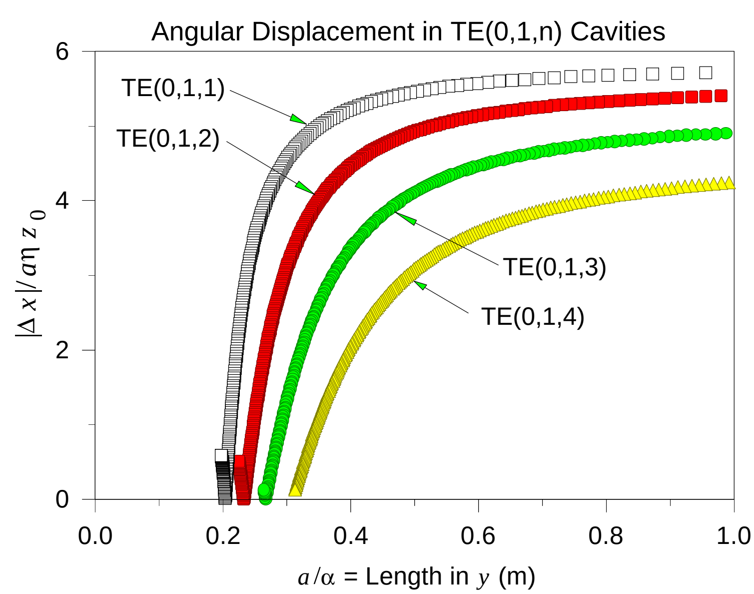

Figure 4: Transverse displacement divided by the cavity length in 800-MHz rectangular $TE_{0,1,n}$ RF cavities as a function of the cavity height a/α, for $\beta_z = 0.84$.

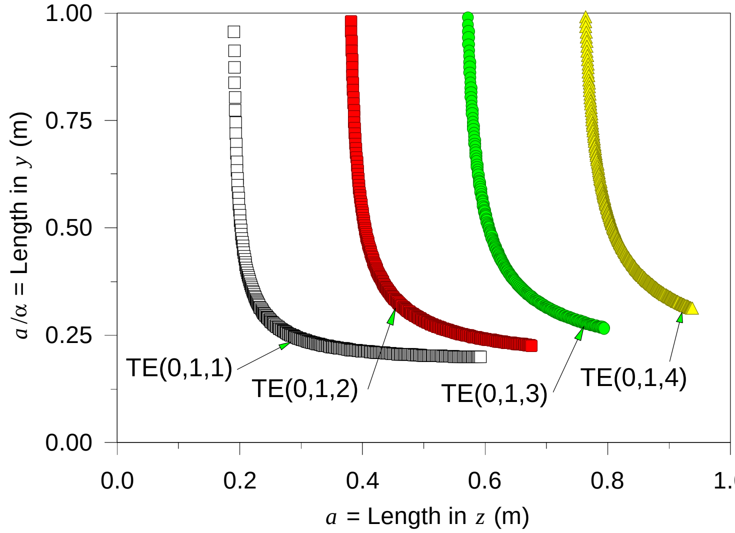

Figure 5: Cavity height a/α as a function of cavity length a.

4 Required Resolution

In the note Princeton/$\mu\mu$/97-4 we discussed how the detector resolution should be at least as small as the quantity we wish to measure. In the case of bunch timing we anticipate that the rms width to be measured is about 40 psec. Taking eq. (63) as representative of the performance of an RF timing cavity, we need to resolve angular displacements of only 1.2 mrad, and better if possible. This puts a limit on the amount of material in the wall of the cavity and in the tracking devices of

$$X_0 < \left(\frac{0.0012 \cdot 165 \cdot 0.84}{15}\right)^2 = 0.00012 \text{ radiation lengths.} \tag{64}$$

Thus the entrance and exit walls of a copper RF cavity should be less than 1.7 μm thick, or less than 43 μm thick if the walls are made of beryllium.

It does not help to stack several timing cavities together, since we are limited by angular resolution, not spatial resolution.

5 Appendix: Displacements in Cavities Phased by 90°

We include this for the record, although it is not useful for the present application.

5.1 Square $\text{TE}_{0,1,1}$ Cavity

The fields in a square $\text{TE}_{0,1,1}$ cavity are now

$$E_x = E_0 \cos\frac{\pi y}{a} \cos\frac{\pi z}{a} \sin\omega t,$$

$$
\begin{aligned}
E_y = E_z &= 0, \\
B_x &= 0, \\
B_y &= -\frac{E_0}{\sqrt{2}} \cos\frac{\pi y}{a} \sin\frac{\pi z}{a} \cos\omega t, \\
B_z &= +\frac{E_0}{\sqrt{2}} \sin\frac{\pi y}{a} \cos\frac{\pi z}{a} \cos\omega t,
\end{aligned} \tag{65}
$$

To the first approximation,

$$
\frac{dv_x}{dt} \approx \frac{eE_0}{2\gamma m} \left\{ \frac{\sqrt{2}+\beta_z}{\sqrt{2}} \cos[(\omega+\omega')t+\epsilon] + \frac{\sqrt{2}-\beta_z}{\sqrt{2}} \cos[(\omega-\omega')t-\epsilon] \right\}, \tag{66}
$$

using the notation of eqs. (15-16). The velocity kick is

$$
\begin{aligned}
\Delta v_x \approx & -\frac{eE_0}{2\gamma m\omega} \Big\{ \cos[(\omega+\omega')t\mathrm{max}+\epsilon] + \cos[(\omega-\omega')t_{\mathrm{max}}-\epsilon] \\
& + \cos[(\omega+\omega')t_{\mathrm{min}}+\epsilon] - \cos[(\omega-\omega')t_{\mathrm{min}}-\epsilon] \Big\},
\end{aligned} \tag{67}
$$

and the transverse displacement is

$$
\begin{aligned}
\Delta x \approx & -\frac{eE_0}{2\gamma mw} \Big\{ \frac{\sqrt{2}}{(\sqrt{2}+\beta_z)\omega} \left(\sin[(\omega+\omega')t_{\mathrm{max}}+\epsilon] + \sin[(\omega+\omega')t_{\mathrm{min}}+\epsilon]\right) \\
& + \frac{\sqrt{2}}{(\sqrt{2}-\beta_z)\omega} \left(\sin[(\omega-\omega')t_{\mathrm{max}}-\epsilon] - \sin[(\omega-\omega')t_{\mathrm{min}}-\epsilon]\right) \\
& - \left(\cos[(\omega+\omega')t_{\mathrm{min}}+\epsilon] - \cos[(\omega-\omega')t_{\mathrm{min}}-\epsilon]\right)(t_{\mathrm{max}}-t_{\mathrm{min}}) \Big\}.
\end{aligned} \tag{68}
$$

Equations (21-31) hold in the present case as well, so we find

$$
\Delta x = \frac{\sqrt{2}\beta_z}{\pi\gamma} \eta\lambda \cos\frac{\sqrt{2}\pi}{2\beta_z}, \tag{69}
$$

where $\lambda = 37.5$ cm is the wavelength at 800 MHz. Thus Δx is independent of the position z_0 of the muon within the bunch. Again

$$
\Delta v_x = 0. \tag{70}
$$