

# Bunch-Timing Measurement in the Muon Cooling Experiment Via a Square $\text{TM}_{1,1,0}$ or a Circular $\text{TM}_{0,1,0}$ RF Accelerating Cavity

## 1 Introduction

An alternative scheme to determine the longitudinal position of a muon with the bunch is to measure its momentum, accelerate it in a cavity phased to null at the center of the bunch, and measure the momentum again. The momentum difference is then proportional to the longitudinal position. This scheme gives results that are independent of the amount of multiple scattering in the RF cavity. Straggling will not present a problem either. The required momentum resolution is comparable to that desired for the momentum measurement by itself. It would be even more advantageous to use a 4-cell accelerating cavity of the same design as for the FOFO cooling section, but phased by  $90^\circ$ .

## 2 Square $\text{TM}_{1,1,0}$ Cavity Fields

The rf cavity is centered on  $(x, y, z) = (0, 0, 0)$ , and is a rectangular box of length  $a$  in  $x$  and  $y$  and length  $b$  along the beam direction  $z$ .

The trajectory of a typical beam particle for the cavity field OFF is parametrized as

$$\begin{aligned}x &= x_0 + \beta_x ct, \\y &= y_0 + \beta_y ct, \\z &= z_0 + \beta_z ct,\end{aligned}\tag{1}$$

where  $c$  is the speed of light. The beam axis is the  $z$ -axis:

$$\beta_x, \beta_y \ll \beta_z, \quad \text{and} \quad \beta_z \approx \beta.\tag{2}$$

We will make the impulse approximation that the cavity fields do not affect the muon trajectories in  $y$  or  $z$ , but only in  $x$ . Thus we assume the  $y$  and  $z$  parametrizations in (1) also hold when the field is ON.

The particle is within the cavity during the interval

$$[t_{\min}, t_{\max}] = \left[ -\frac{b}{2\beta_z c} - \frac{z_0}{\beta_z c}, \frac{b}{2\beta_z c} - \frac{z_0}{\beta_z c} \right]. \quad (3)$$

The wave equation tells us that for a  $\text{TM}_{1,1,0}$  cavity

$$\frac{\omega}{c} = \sqrt{2} \frac{\pi}{a}, \quad \text{so} \quad a = \frac{\sqrt{2}}{2} \lambda = 0.707 \lambda. \quad (4)$$

where  $\omega$  is the angular frequency and  $\lambda = c/2\pi\omega$  is the wavelength. For frequency  $\nu = 800$  MHz,  $\lambda = 37.5$  cm.

The cavity is phased so that the electric field is zero at  $t = 0$ . In Gaussian units,

$$\begin{aligned} E_x = E_y &= 0, \\ E_z &= E_0 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} \sin \omega t, \\ B_x &= -\frac{E_0}{\sqrt{2}} \cos \frac{\pi x}{a} \sin \frac{\pi y}{a} \cos \omega t, \\ B_y &= \frac{E_0}{\sqrt{2}} \sin \frac{\pi x}{a} \cos \frac{\pi y}{a} \cos \omega t, \\ B_z &= 0. \end{aligned} \quad (5)$$

### 3 Energy Gain: Leading Approximation

The gain in energy,  $\Delta U$ , when a muon traverses the cavity is

$$\begin{aligned} \Delta U &= e \int E_z dz = e\beta_z c \int_{t_{\min}}^{t_{\max}} E_z dt = e\beta_z c E_0 \int_{t_{\min}}^{t_{\max}} \cos \frac{\pi x(t)}{a} \cos \frac{\pi y(t)}{a} \sin \omega t dt \\ &\approx e\beta_z c E_0 \int_{t_{\min}}^{t_{\max}} \sin \omega t dt = \frac{\beta_z c e E_0}{\omega} (\cos \omega t_{\min} - \cos \omega t_{\max}) \\ &\approx -\frac{\beta_z c e E_0}{\omega} \frac{2\omega z_0}{\beta_z c} \sin \frac{\omega b}{2\beta_z c} = -2eE_0 z_0 = -2eE_0 \beta_z c \Delta t, \end{aligned} \quad (6)$$

using eq. (3) and supposing that the transverse size of the beam is small compare to  $a$ . In eq. (6) we have put  $b = \beta_z \lambda / 2$  to maximize  $\Delta U$ .

From eq. (6) we see that  $\Delta U$  depends on the  $x$  and  $y$  positions and slopes only in second order.

For a peak field of  $E_0 = 40$  MV/m, and 165 MeV/ $c$  muons for which  $\beta_z = 0.84$ , we have

$$\Delta U = 0.02 \text{ [MeV]} \left[ \frac{\Delta t}{1 \text{ ps}} \right]. \quad (7)$$

Since  $U \Delta U = c^2 P \Delta P$ , the energy gain corresponds to a relative momentum change

$$\frac{\Delta P}{P} = \frac{2eE_0 \beta_z c \Delta t}{\beta_z c P} = \frac{2 \cdot 40 \text{ [Mv/m]} \cdot 3 \times 10^{-4} \text{ [m/ps]} \cdot \Delta t \text{ [ps]}}{165 \text{ [MeV/c]} \cdot c} = 0.00014 \left[ \frac{\Delta t}{1 \text{ ps}} \right]. \quad (8)$$

## 4 Cylindrical $\text{TM}_{0,1,0}$ Cavity

The wave equation tells us that for a cylindrical  $\text{TM}_{0,1,0}$  cavity of radius  $a$ , diameter  $d$  and length  $b$  along the  $z$ -axis,

$$\frac{\omega}{c} = \frac{2.405}{a}, \quad \text{so} \quad d = 2a = \frac{2.405}{\pi} \lambda = 0.766\lambda. \quad (9)$$

Thus the diameter of the cylindrical  $\text{TM}_{0,1,0}$  cavity is slightly larger than the edge of a square  $\text{TM}_{1,1,0}$  cavity, but less than the diagonal of the square cavity. So in practice the cylindrical cavity is to be preferred.

The fields in the cylindrical cavity whose electric field vanishes at  $t = 0$  are

$$\begin{aligned} E_r = E_\phi &= 0, \\ E_z &= E_0 J_0 \left( \frac{2.405r}{a} \right) \sin \omega t, \\ B_\phi &= E_0 J_1 \left( \frac{2.405r}{a} \right) \cos \omega t, \\ B_r = B_z &= 0. \end{aligned} \quad (10)$$

In the approximation that the muons are close to the  $z$ -axis,  $J_0(2.405r/a) \approx 1$ , and the results of sec. 3 hold for the cylindrical  $\text{TM}_{0,1,0}$  as well as for the square  $\text{TM}_{1,1,0}$  cavity.

## 5 Discussion

As in our note Princeton/ $\mu\mu$ /97-6, we suppose the timing resolution function is well understood and so we relax the requirement on timing resolution to, say  $\sigma_{t,D} = 0.5\sigma_t = 20$  ps, following Table 1 of our note Princeton/ $\mu\mu$ /97-4. This requires knowing the timing resolution function to 4%. Then the momentum resolution of the detector should be  $\sigma_{P,D}/P = 0.0028$ .

A configuration with two momentum spectrometers surrounding an acceleration cavity is favorable for the required study of the resolution functions of the momentum spectrometers and the associated tracking systems. With the cavity off, one spectrometer can be used to characterize the other. Complete characterization of the timing measurement will still require comparison between the devices upstream and downstream of the cooling section.

Also, it would be advantageous to use a 4-cell accelerating cavity of the same design as for the FOFO cooling section. Present parameters for these 4-cell structures are a length of  $b = \lambda\beta_z/3 = 10.5$  cm per cell, and inner radius 14.6 cm. Because each cell is only 2/3 of the length we assumed in eq. (6) the timing correlation per cell is reduced by  $\sin(60^\circ) = 0.866$ . The total correlation over 4 cells is then

$$\frac{\Delta P}{P} = 4 \cdot 0.87 \cdot 0.00014 = 0.0005 \left[ \frac{\Delta t}{1 \text{ ps}} \right]. \quad (11)$$

We might now strive for a timing resolution of  $8 \text{ ps} = 0.2\sigma_t$ , in line with our general goal for all six phase-space parameters.

An advantage of the timing measurement via an accelerating cavity is that multiple scattering within the RF cavity does not affect the time resolution. However, the timing

resolution will be degraded by straggling in the cavity walls and surrounding detectors. An approximate expression for momentum straggling [Rossi, *High Energy Particles*, p. 31, eqs. (8) and (9) with  $E'_{\max} \approx 2\gamma^2\beta^2m_e c^2$ ] is

$$\sigma_{P,\text{straggling}} = \frac{\gamma r_e m_e c^2}{\beta c} \sqrt{2\pi N_0 \frac{Z}{A} s \left(1 - \frac{\beta^2}{2}\right)} \approx 0.11 \text{ [MeV}/c] \sqrt{\left[\frac{s}{1 \text{ g/cm}^2}\right]}, \quad (12)$$

where  $r_e$  is the classical electron radius,  $N_0$  is Avagadro's number and  $s$  is the amount of material. The numerical result holds for  $Z/A = 1/2$ , and  $\beta = 0.84$ .

The momentum smearing due to straggling should be small compared to the desired resolution, *i.e.*, small compared to  $\sigma_{P,D} = 8 \cdot 0.0005P = 0.66P \text{ MeV}/c$  for  $P = 165 \text{ MeV}/c$ . Then the total material in the 5 walls of the 4-cell structure plus surrounding chambers must satisfy

$$s < \left(\frac{\sigma_{P,D}}{0.11}\right)^2 = \left(\frac{0.66}{0.11}\right)^2 = 36 \text{ [g/cm}^2\text{]}. \quad (13)$$

This places very little constraint on the choice of materials.