

Extension into Three Dimensions of the Classical Problem
of Apollonius

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(Figures 17-20 are the solid models made of plastic)

Topic Outline

I. Introduction

II. The Apollonian Method

- A. The radical center of two and three circles.
- B. The homothetic centers of two circles.
- C. Solution to the Problem of Apollonius by the Apollonian method.

III. Gergonne's Solution

- A. The radical center of three circles.
- B. The lines containing a triad of homothetic centers.
- C. Poles and polars.
- D. Solution to the Problem of Apollonius by Gergonne's solution.

IV. The spheres tangent to four given spheres.

- A. The statement of the extended problem of Apollonius.
- B. The planes formed by the triads of homothetic centers of four spheres.
- C. The poles of the above planes with respect to the four given spheres.
- D. The radical center of four spheres.
- E. The construction for the spheres tangent to four given spheres.

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Introduction

The title of this project is "An Extension of the Classical Problem of Apollonius into the Third Dimension." The Problem of Apollonius is one of the classical Greek era, being to construct a circle tangent to three other circles. I first encountered the Problem of Apollonius in The World of Mathematics.

I first tried some hit or miss solutions, which almost resulted in a true solution. Then I obtained some college geometry books which gave the solution to the problem of Apollonius. After studying these, I wondered if this problem might be extended from the second to the third dimension. I could find no information on this subject, so, as far as I know, this problem is original.

The first part of this project is dedicated to the Problem of Apollonius as it is necessary background information. The last part of the paper is then my original work on the three dimensional problem. This section is smaller in volume as it is the culmination of the previous work which is not repeated.

The final result of this work is the construction of the points of tangency of a sphere tangent to four other spheres. I have used the plastic models to demonstrate the tetrahedron formed by the centers of the spheres and the various points related to the sections of the spheres formed by the faces of the tetrahedron.

P A R T I

The Apollonian Method

The Problem of Apollonius is to construct a circle tangent to three other circles. The solutions given here will deal with cases where the three circles are mutually external, of unequal radii, and centers which are non-collinear. Given this information, there will be eight possible circles tangent to the other three.

There are many solutions to this problem, but I will demonstrate only the two which have the greatest reference to my problem. The first one that I will present is similar to the solution of Apollonius himself, but with updated terms. This solution demonstrates the basic concepts of the problem, and it shows a way of ~~the~~ solving the problem with them. The second is a more modern one called "ergonne's Solution, which I will use in solving the three dimensional problem.

In the Apollonian Method, there are three steps to the solution starting with Problem I.

Problem I: Construct a circle passing through two given points and tangent to a given circle.

To solve this problem, we must first establish the radical axis of two circles. The definition of a radical axis is the locus of points from which equal tangents may be drawn to the two circles.

Case I is when the two circles intersect. When the circles intersect, the radical axis is the secant of their common chord. Let the two intersecting circles be (C_1) and (C_2) (Fig. 1). The chord AB produced will be the common secant of the two circles. Let any point on this secant be P,

and construct tangent PM to circle (C_1) . With PM as radius, construct circle (P) intersecting circle (C_1) at M and N and Circle (C_2) at O and C . The tangents PM , PN , PO , and PC are equal as they are all the mean proportional between secant PA and its external segment PB . A similar proof holds true when the circles are tangent.

It was shown above that the radical axis of two intersecting circles is the common chord of the two circles and this chord is perpendicular to the line of centers of the two circles. From this, the theorem is derived that the radical axis of any two circles is perpendicular to the line of centers of the two circles.¹

Case II deals with proving the theorem that the radical axes of three given circles with non-collinear centers are concurrent. Let the given circles be (C_1) , (C_2) and (C_3) (Fig. 2). The radical axis of circles (C_1) and (C_2) is line AE and the radical axis of circles (C_2) and (C_3) is line DF . These two lines intersect at point P . The tangent PL to circle (C_1) will equal tangent PM to circle (C_2) , and tangent PO to circle (C_3) will equal tangent PM . Therefore, PO equals PL , and all the tangents from P to the three circles are equal. Point P is called the radical center of the three circles.

Case III. Case II (Fig. 2) shows that the tangents from P to circles (C_1) and (C_3) are equal. Therefore, it is evident that P is on the radical axis of circles (C_1) and

¹-----
A more complete discussion of this theorem may be found in Modern College Geometry by Davis, pp. 30-33

(C_3). Since the radical axis of two circles is a line perpendicular to the line of centers, the radical axis of circles (C_1) and (C_3) is the perpendicular from F to line C_1C_3 .

To construct the radical axis of two mutually external circles, first draw a third circle intersecting them both. Then draw the two common secants, and where they intersect, drop a perpendicular to the line of centers of the two circles. This line will be the radical axis of the two circles.

With these theorems and constructions, it is now possible to solve Problem I. The circle (C_1) and the points P_1 and P_2 are given (Fig. 3). Since the required circle must pass through P_1 and P_2 , the center must be on the perpendicular bisector of line P_1P_2 . Next, using any point C_2 of line AD , the perpendicular bisector of P_1P_2 , construct circle (C_2) passing through point P_1 and P_2 , and intersecting circle (C_1). The common secant QS of circles (C_1) and (C_2) will be their radical axis. Since the circle (C_2) and the required circles (C_3) and (C_4) will pass through points P_1 and P_2 , line P_1P_2 produced will be the radical axis of those two circles. The intersection of P_1P_2 and QS will determine point R , the radical center of all four circles.

Since the radical axes of the required circles through R will be perpendicular to the line of centers of these circles, and the circles will be tangent, the tangents from R to circle (C_1) will determine the points of tangency of the required circles. These points, T_1 and T_2 drawn through point C_1 intersect line AD at points C_3 and C_4 , the centers of the

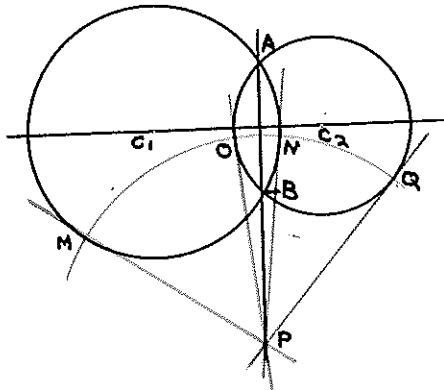


Fig. 1

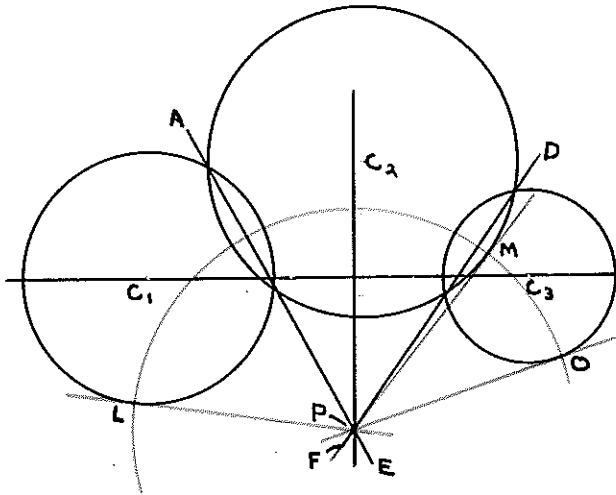


Fig. 2

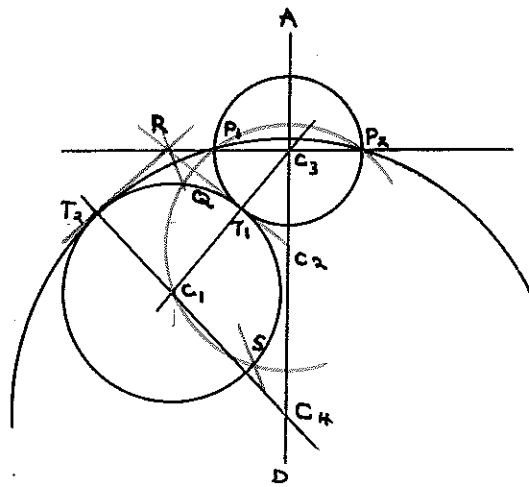


Fig. 3

required circles.

Problem II: Construct a circle tangent to two mutually external circles that passes through a given point outside the circles.

To solve this problem by the Apollonian method, a discussion of homothetics is necessary now. Homothetic is defined as "similar and similiary placed". Two figures may be either directly or inversely homothetic. The homothetic figures to be used in the following constructions will all be circles with positive radii and therefore all directly homothetic.

Any two circles are homothetic as they are all similar in shape and remain in the same relation to each other throughout rotation. All homothetic figures have a homothetic ratio and that of two circles is the ratio of their radii. Any two circles have two homothetic centers which divide their line of centers internally and externally in the ratio of their radii.

Given two circles (C_1) and (C_2), draw their line of centers and produce it (Fig. 4). Then draw any two parallel radii, C_1A_1 and C_2A_2 on the same side of line C_1C_2 and produce line A_1A_2 until it meets line C_1C_2 at E, the external homothetic center. This makes two similar triangles. Next, produce line C_2A_2 until it meets circle (C_2) at A_3 . Line A_1A_3 intersects line C_1C_2 at I, the internal homothetic center. This also forms two similar triangles. From the

similar triangles C_1A_1E and C_2A_2E , this equation is obtained:

$$\frac{EC_1}{EC_2} = \frac{EA_1}{EA_2} = \frac{r_1}{r_2} = k$$

when r_1 is the radius of circle (C_1) and r_2 is the radius of circle (C_2) and k is the homothetic ratio.

From the similar triangles C_1A_1I and C_2A_2I , a similar equation holds true, showing that I divides the line of centers internally in the ratio of the radii. It also noted that the common internal and external tangents to the two circles intersect the line of centers at the internal and external homothetic centers respectively, as they are also determined by parallel radii.

The above construction holds true for circles be they tangent, mutually external, intersecting, or one inside the other as long as they are of unequal radii. If the circles are concurrent, the homothetic centers are infinity and the real center of the circle.

Homothetic centers are sometimes referred to as centers of similitude.

Also in Figure 4, it is noted that line EA_2A_1 intersects circle (C_2) at B_2 and A_2 and circle (C_1) at B_1 and A_1 . The pairs of points B_1 and B_2 , and A_1 and A_2 are said to be anti-homologous. The same holds true for the pairs of points A_1 and A_3 and D_1 and D_2 . Any line drawn from either the internal or external homothetic center intersecting both circles will produce this relationship.

To get a better picture of the solution to Problem II,

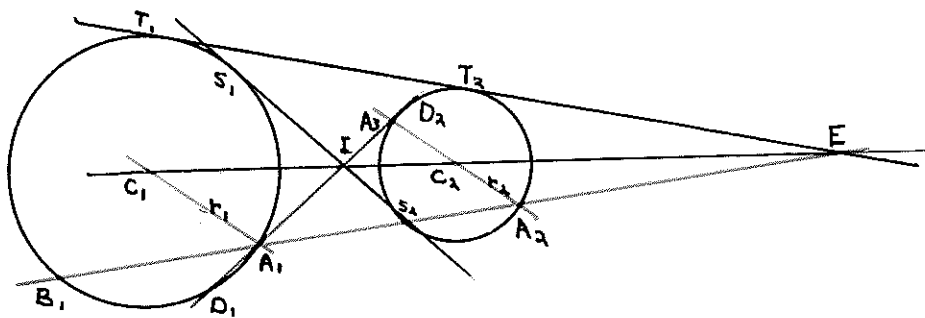


Fig. 4

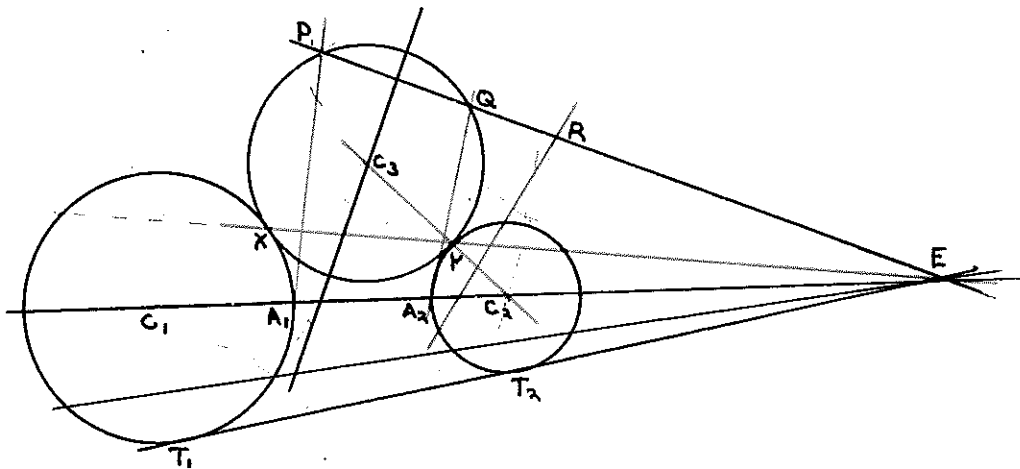


Fig. 5

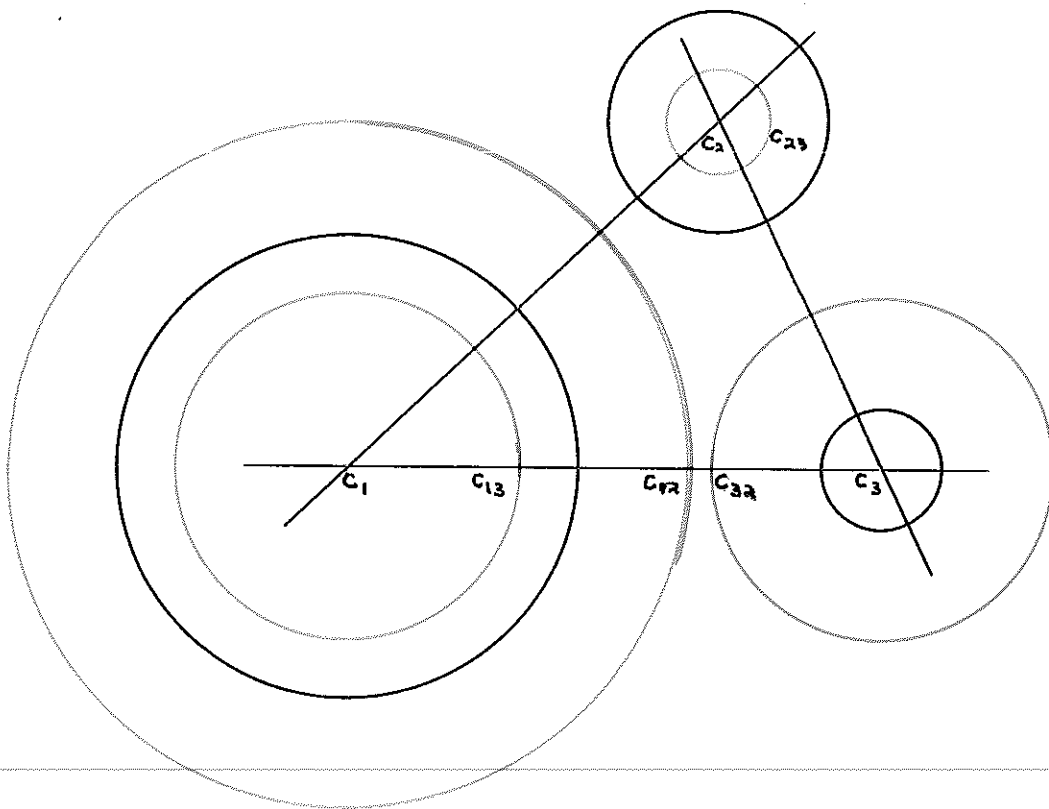


Fig. 6

it is first necessary to take the solved problem and work backwards to get the solution. The two given circles are (C_1) and (C_2) and the required circle is (C_3) (Fig. 5). The lines of centers C_1C_3 and C_2C_3 intersect the circles at the points of tangency X and Y . The line XY produced intersects the line of centers C_1C_2 at E , the external homothetic center. Therefore X and Y are anti-homologous. The common external tangent T_1T_2 is drawn from E to circles (C_1) and (C_2) . Point P is the given point and Q is the intersection of circle (C_3) and line EP . Points A_1 and A_2 are the intersections of line C_1C_2 with their respective circles internally and therefore, they are anti-homologous.

From the theorem² that the product of the lines determined by two anti-homologous points and a homothetic center equals the product of the tangents from the same homothetic center to the two circles, this equation is obtained:

$$EQ \cdot EP = EX \cdot EY = EA_1 \cdot EA_2 = ET_1 \cdot ET_2$$

Line EQ is the unknown and points A_1 and A_2 are the most readily usable. Therefore, point Q is found by constructing the angle EA_2Q equal to angle EPA_1 , because of similar triangles. Point Q having been found, the given is the same as in Problem I, which is used to complete the problem. The circle tangent to one of the circles will be tangent to the other also as Q is a function of both circles.

Problem III: Construct a circle tangent to three given circles.

²Modern College Geometry by Davis, pp. 93-94

The three given circles are (C_1) , (C_2) and (C_3) (Fig.6). Since Problem II provides a solution to a circle tangent to two circles and passing through a given point, one of the three circles must be reduced to a point, with the other radii corrected for this. This is done by either adding or subtracting one radius from the other two circles.

The first case is when the radius of one circle is subtracted from the other two. As the circles must all have positive radii, only the smallest radius may be subtracted from the other two, this being circle (C_3) in Figure 6. With this construction completed, there are circles concurrent to (C_1) and (C_2) inside them labeled (C_{13}) and (C_{23}) respectively. It is seen then, that any circle tangent to circles (C_{13}) and (C_{23}) and passing through point C_3 will have the same center as the circle that is tangent to the three given circles.

The completed construction is seen in Figure 7. The given circles are (C_1) , (C_2) and (C_3) , and the required circles (C_4) and (C_5) . The radius of (C_3) has been subtracted from (C_1) and (C_2) leaving circles (C_{13}) and (C_{23}) , and point C_3 . The rest of the construction is carried out without letters, but it is the same as Problems I and II. This construction supplies the completely internally and externally tangent circles, (C_4) and (C_5) .

The construction of Figure 7 supplies two of the eight solutions, leaving six to be supplied by the cases where the radii are added. Figure 6 also shows the construction for adding the radii. The radius of circle (C_2) is added to the

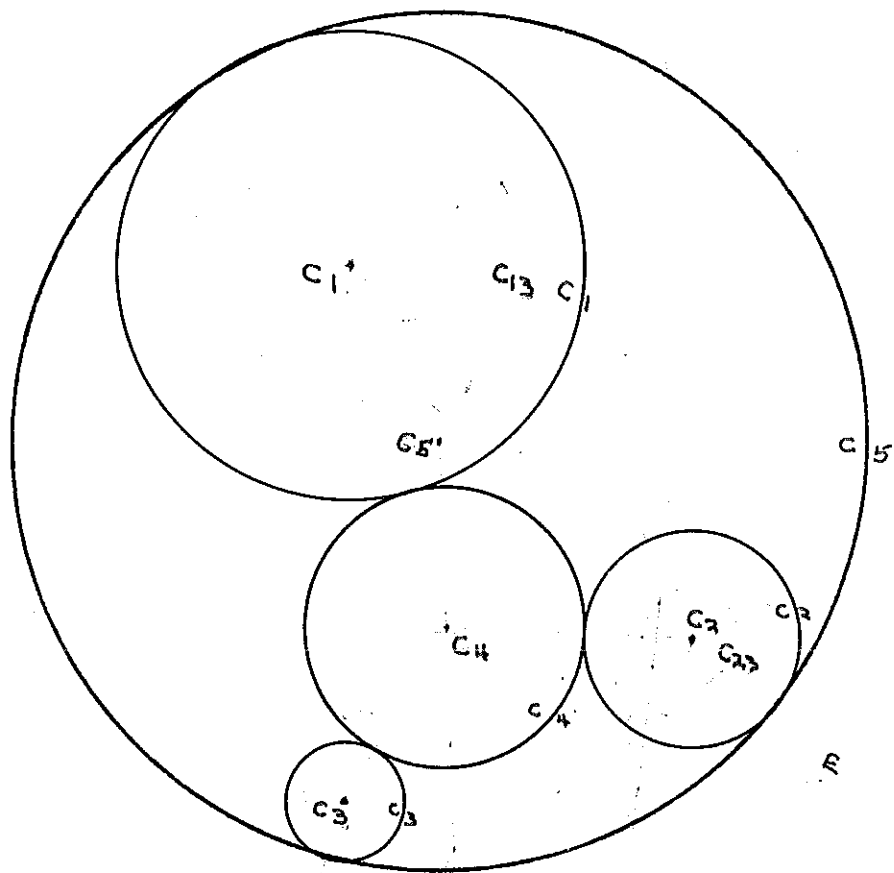
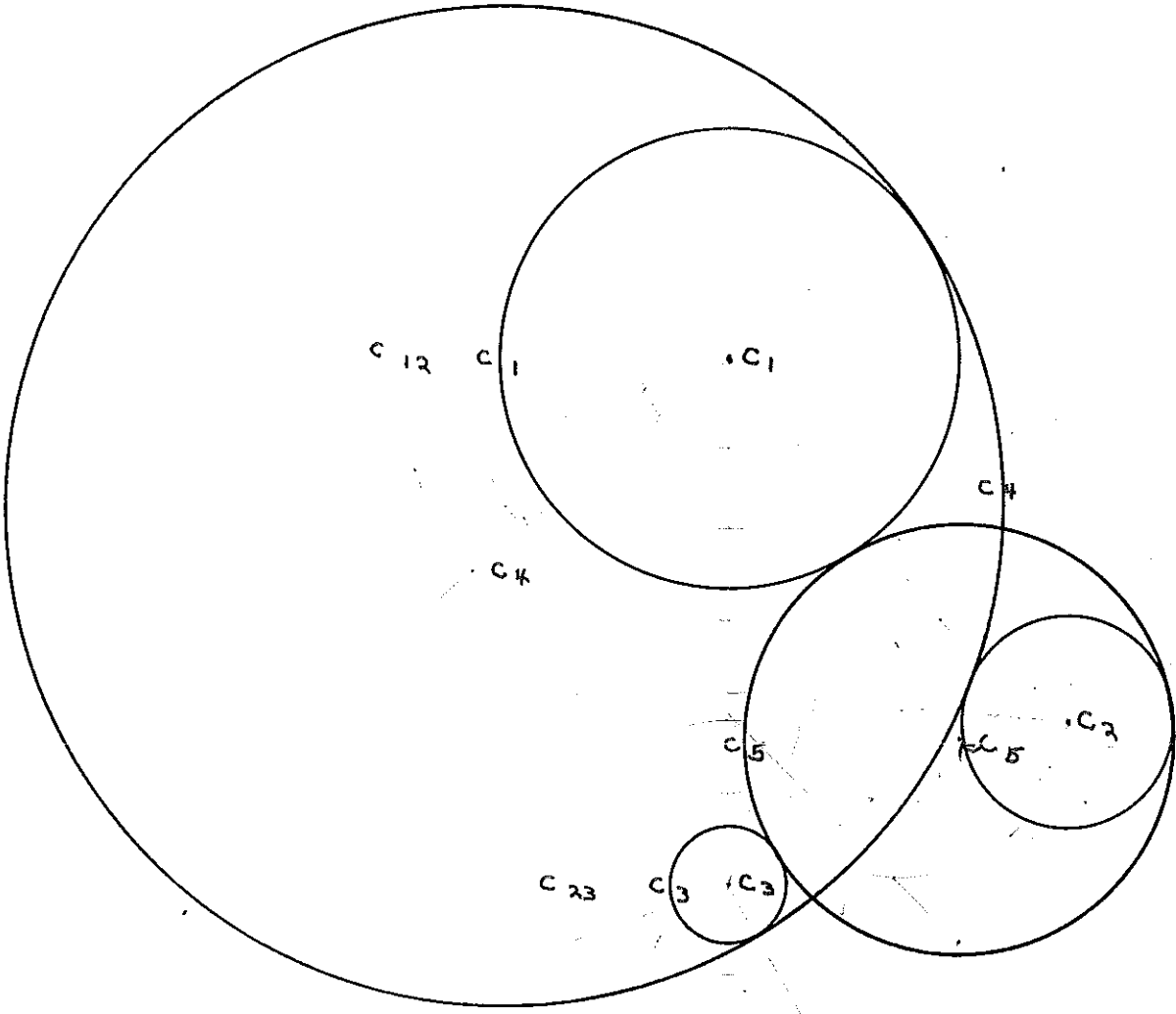


Fig. 7

others making the large circles (C_{12}) and (C_{32}) . This solution will supply the circle tangent internally to (C_2) and externally to (C_1) and (C_3) , and a circle tangent externally to (C_2) and internally to the other two.

The completed construction is seen in Figure 8. The given circles are (C_1) , (C_2) and (C_3) and the required circles (C_4) and (C_5) . The circles with the added radii are (C_{12}) and (C_{32}) .

The other four circles may be constructed by adding the radii of circles (C_1) and (C_3) to the other two circles respectively.



E
Fig. 8

Gergonne's Solution

The second solution to the problem of Apollonius is Gergonne's solution, the neatest and most famous of all. Gergonne's construction is: "Find the poles with respect to the given circles of a line containing a triad of homothetic centers of the circles taken in pairs. The lines connecting these poles with the radical center of the three circles will meet the three circles at the points of contact of ^{a pair of} the circles sought." The three circles are mutually external and the pairs of circles found will be made up of two opposites, i.e. internal and external.

From the previous information, it will be easier to work from the back to the front in this construction. To find the radical center of three mutually external circles, first find the radical axis of a pair of them. This construction is reviewed in Figure 9. Then find the radical axis of another pair of them. Where the radical axes meet is the radical center of the three circles. This construction is shown in Figure 10. The three given circles are (C_1) , (C_2) and (C_3) . The two inked lines are the radical axes of two pairs of the circles, and they intersect at R, the radical center.

The construction for the lines containing a triad of homothetic centers is as follows. The three given circles are (C_1) , (C_2) and (C_3) (Fig. 11). Construct the homothetic centers for each pair of circles, i.e. I_{12} and E_{12} for circles (C_1) and (C_2) , etc., until there are three internal and three external homothetic centers. These six points may be connected

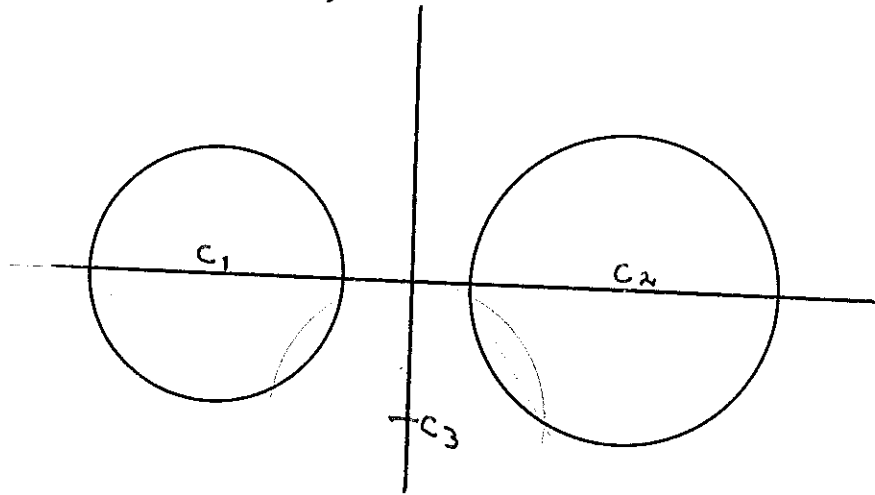


Fig. 9

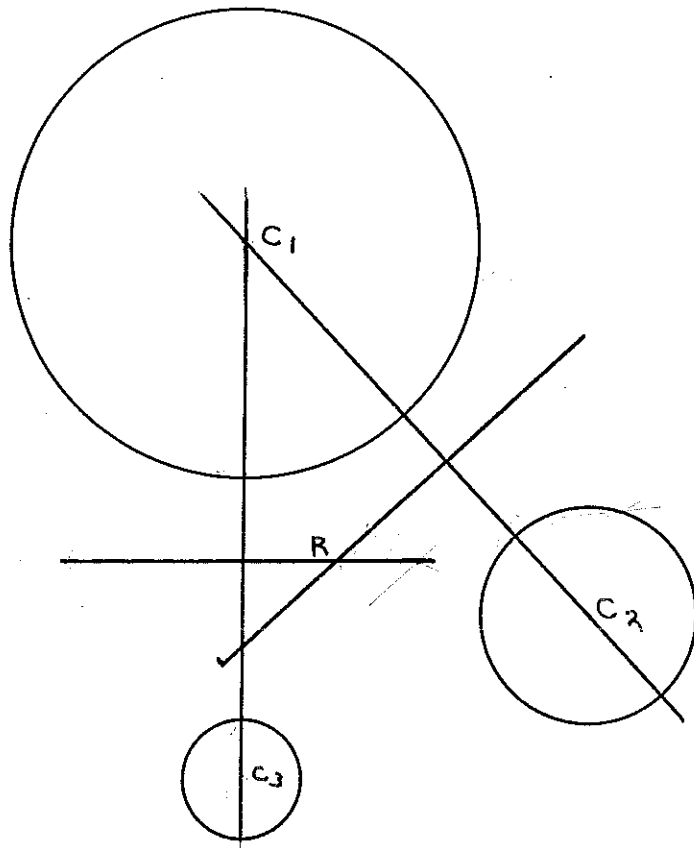


Fig. 10

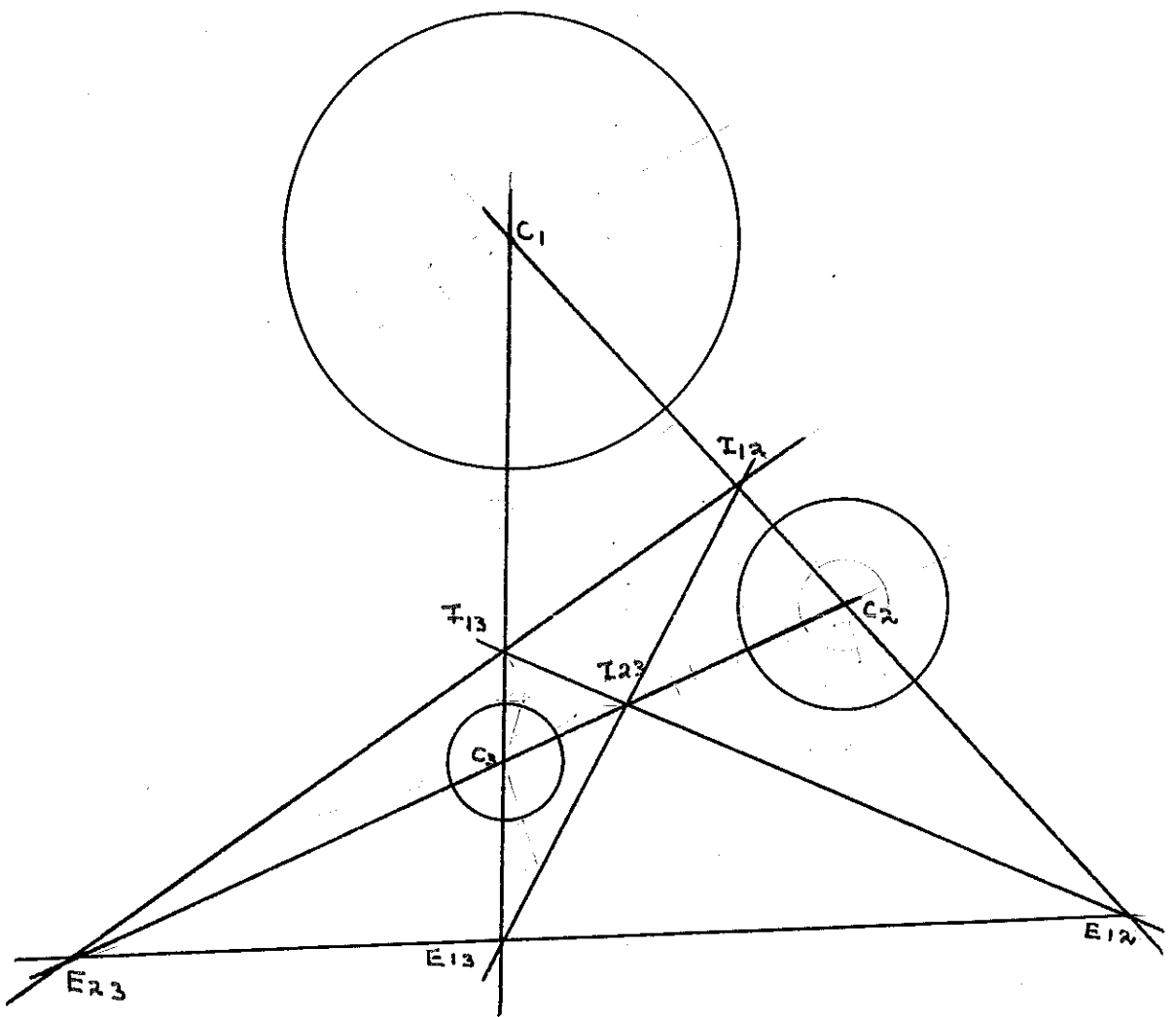


Fig. 11

by four lines, each containing three centers. These lines are the triads of homothetic centers referred to in Gergonne's construction.

A brief summary of poles and polars is necessary before this construction may be completed.³ Given circle (C_1) and point P_1 outside the circle (Fig. 12), find point P_2 on line C_1P_1 inside the circle so that the product of P_1C_1 and P_2C_1 equals the radius of circle (C_1) squared. The perpendicular AB to line P_1C_1 at P_2 is the polar of point P_1 with respect to circle (C_1) , and point P_1 is the pole of line AP_2B with respect to the circle.

To construct the polar of a point with respect to a circle, first draw the line from the given external point, P_1 , to the center of the circle (C_1) (Fig. 13). Then construct the two tangents from P_1 to circle (C_1) , P_1T_1 and P_1T_2 . The line joining points T_1 and T_2 is the polar of P_1 with respect to circle (C_1) . If point P_2 is inside the circle (C_1) as in Figure 12, erect the perpendicular AP_2B to line C_1P_2 . At point A , construct a tangent, and where this tangent intersects line C_1P_2 extended, erect a perpendicular. This perpendicular is the polar of point P_2 with respect to circle (C_1) . It is seen, as point P approaches the center of the circle, its polar approaches being an infinite distance away, and as the point P approaches an infinite distance from the center, its polar approaches the center.

³A more complete discussion may be found in Collge Geometry by Altshiller-Court, pp. 148-151

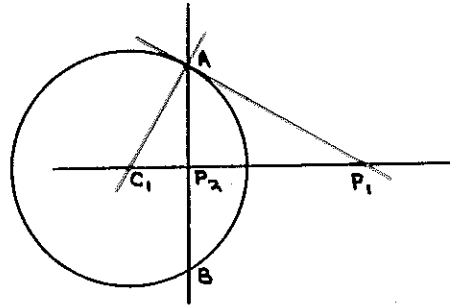


Fig. 12

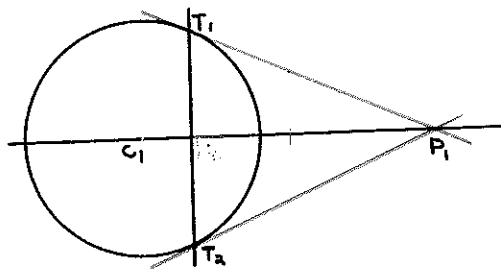


Fig. 13

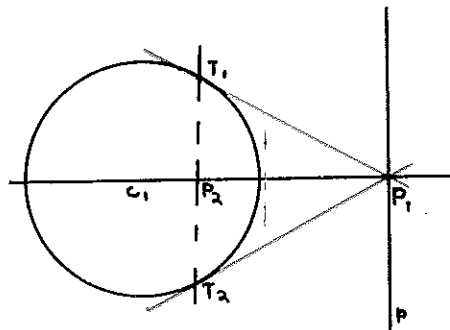
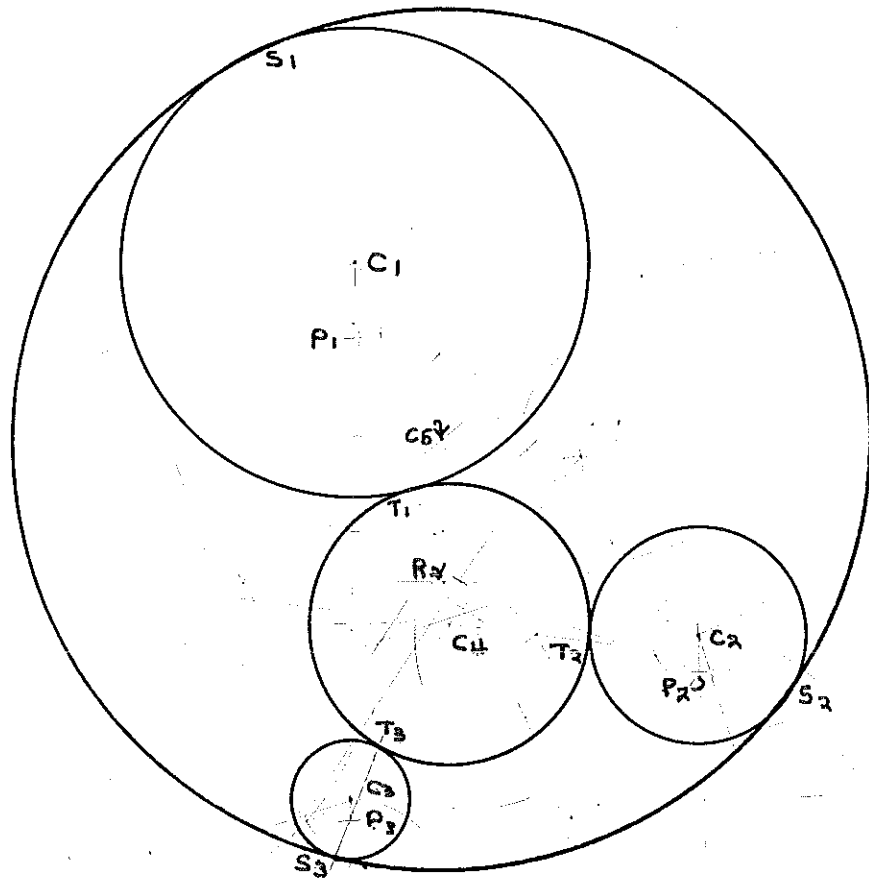


Fig. 14

To construct the pole of a line with respect to a circle, construct the perpendicular from the center of the circle, (C_1) , to the line p (Fig. 14). At the point of intersection, P_1 , construct the tangents to the circle, P_1T_1 and P_1T_2 . The intersection of the line T_1T_2 with line C_1P_1 at P_2 is the pole of line p with respect to circle (C_1) . If line p is inside the circle, P_2 the pole, will be the intersection of the two tangents to the circle at the points that line p intersects the circle. Again it is to be noted that as the line approaches an infinite distance from the circle, the pole approaches the center, and as the line approaches the center, the pole approaches being an infinite distance away, until, when the line is a diameter, there are two poles, each an infinite distance away.

With the construction for Figure 14, it is possible to construct Apollonius's Solution. The three given circles are (C_1) , (C_2) and (C_3) , and the required circles are (C_4) and (C_5) (Fig. 15). As only one pair of circles is to be found in the first construction, only one of the lines containing a triad of homothetic centers will be used. In this case, it is the one with the three external homothetic centers, E_{12} , E_{13} , and E_{23} . Then, drop the perpendiculars to line $E_{12}E_{23}$ from the centers of the three given circle, and construct the three poles, P_1 , P_2 , and P_3 with respect to their respective circles. Construct the radical center of the three circles, R , and connect it with the three poles to get the



E23

E13

E12

Fig. 15

points of contact T_1 , T_2 , T_3 , and S_1 , S_2 , and S_3 . These are the points of tangency of the required circles. The two required circles will be circumscribed around triangles $T_1T_2T_3$ and $S_1S_2S_3$. The line of centers of these two circles will pass through R and be perpendicular to $E_{12}E_{23}$.

In Figure 16, another case is shown. The line of homothetic centers used is $E_{23}I_{13}I_{12}$. This arrangement carried on will determine a circle tangent externally to circles (C_2) and (C_3) and internally to circle (C_1) , and a circle that is the exact opposite. The other two lines of homothetic centers will determine the other four circles.

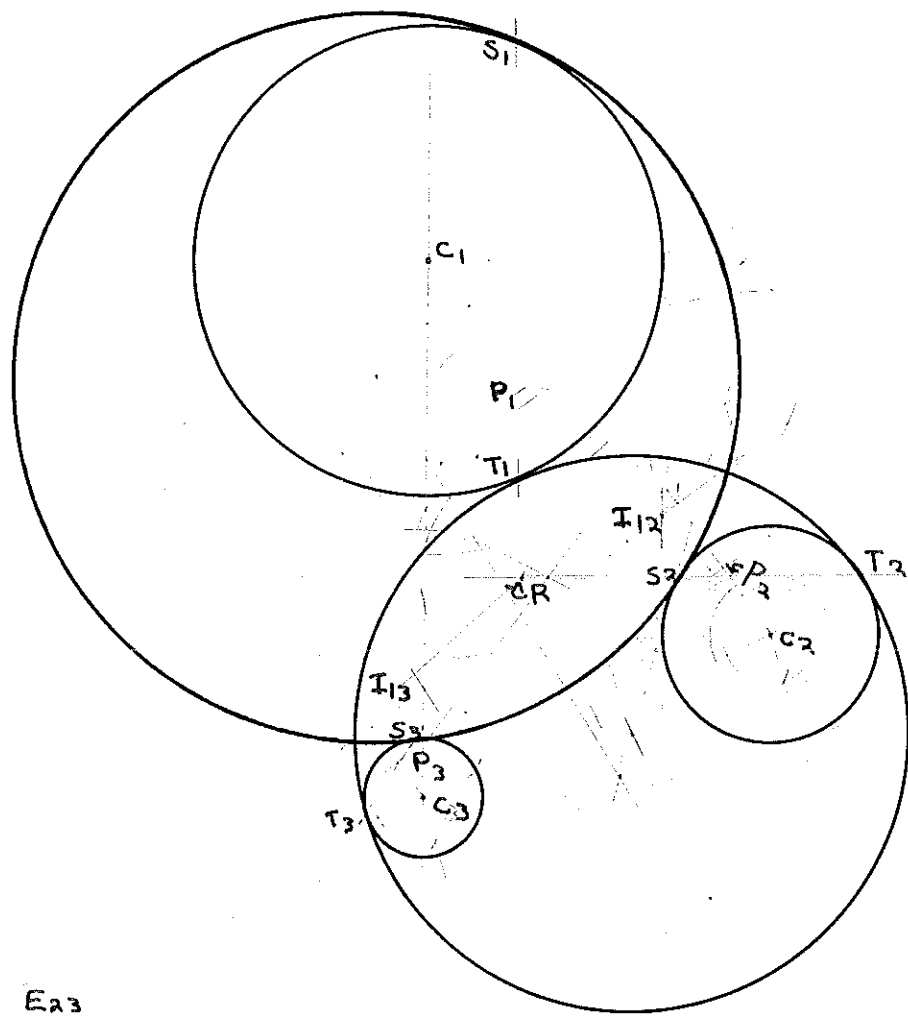


Fig. 16

P A R T II

The Spheres Tangent to Four Given Spheres

The Problem of Apollonius is to construct a circle tangent to three other circles. If, instead of circles, you are given spheres, the problem is considerably changed. If three spheres are given, using Gergonne's Solution, a line perpendicular to the plane of the centers of the spheres may be found which is the locus of the centers of the spheres tangent to the other three. Therefore, a fourth sphere is necessary to give the problem meaning.

The three dimensional problem is then, construct a sphere tangent to four other spheres. If the spheres are mutually external, their centers not coplanar, and of unequal radii, there will be a maximum of sixteen solutions. In solving this problem, Gergonne's Solution will be used. Because a new dimension has been added, the construction needs to be altered accordingly. The resulting construction is this: "Find the planes formed by the intersections of the lines containing a triad of homothetic centers of the four spheres taken in pairs. Then find the poles of one of these planes with respect to the four spheres. Connect these four poles with the radical center of the spheres to find the points of contact of a pair of the spheres sought for."

The solution begins with the planes formed by the lines of homothetic centers. The four given spheres are (S_1) , (S_2) , (S_3) , and (S_4) (Fig. 17). By joining the centers of these spheres, a tetrahedron is formed. For ease of construction,

I have used a regular tetrahedron, though the construction holds true for irregular ones also. Each of the faces of the tetrahedron contains three circular sections of the spheres. The first step is to construct the homothetic centers of the sets of circles in the four faces. Because the circles along an edge between any two adjoining faces of the tetrahedron are sections of the same sphere, their homothetic centers will be located at the same place along their line of centers. And also, as these two homothetic centers will be the parts of two separate triads of homothetic centers, their triads meet at a point to determine a plane. Because of the number of the homothetic centers, there are eight of these planes all told.

If Figure 17, only the external homothetic centers are used. From the analogy stated above, there will be a plane formed. The plane is represented by the flat sheet of plastic and called plane P. The tetrahedron is formed by the centers of the spheres labeled (S_1) , (S_2) etc. The homothetic centers are all external and labeled E_{12} , E_{23} etc.

The next part of the problem is to find the poles of the plane found in Figure 17 with respect to the spheres. This is shown in Figure 18. The four given spheres are (S_1) , (S_2) , (S_3) , and (S_4) . The plastic sheet represents the plane P, which, in this case, is determined by only a portion of Figure 17. To find the poles required, drop the perpendiculars from the centers of the spheres to plane P.

The poles are then found by taking any plane containing the perpendicular and using the construction stated in Gergonne's construction. As plane $S_2S_3S_4$ was very nearly perpendicular to plane P , by chance, I have constructed the poles to those spheres right in the plane stated. The resulting poles are labeled P_1 , P_2 , P_3 , and P_4 .

The third part of the problem is to find the radical center of the four given spheres. The spheres are (S_1) , (S_2) , (S_3) , and (S_4) (Fig. 19). The first step is to find the radical center of the spheres in one of the faces of the tetrahedron formed by the centers of the spheres. Face $S_1S_2S_3$ was used first, and point R_{123} , the radical center of the circles in that plane found. The perpendicular at R_{123} to the face was erected. This is the locus of points from which equal tangents may be drawn to the three spheres. Then face $S_2S_3S_4$ was used and a similar construction carried through. Point R_{234} is the radical center of the face and the perpendicular through it is also erected. As this line is the locus of points from which equal tangents may be drawn to the three spheres in that face, the intersection of the two perpendiculars through the radical centers will be the radical center of all four spheres.

With these constructions, it is possible to solve the problem. The four given spheres are (S_1) , (S_2) , (S_3) , and (S_4) (Fig. 20). Figure 20 is a combination of Figures 18 and 19. The radical center is found by using the method given in Figure 19 using faces $S_1S_2S_4$ and $S_1S_2S_3$. The four poles

are found as shown in Figure 18. The last step is to connect the radical center with these poles. Each of the four lines will intersect its sphere in two points, I for the points of contact for the completely internal sphere, and E for the points of contact of the completely external one. The spheres circumscribed about the two tetrahedrons formed will be the required ones.

Each of the seven remaining pairs of spheres may be found by using a similar construction for each of the planes formed by the various lines containing a triad of homothetic centers.

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