Quantum Limit to Radiation Damping of a Charged Oscillator

Zahid M. Hasan and Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(December 23, 2005)

1 Problem

An electron of mass m and charge e moves in an isotropic three-dimensional harmonic potential with "spring constant" K such that its trajectory is nearly circular at all times. What is the characteristic time (time constant) for the decay of the energy of the system due to electromagnetic radiation according to a classical analysis?

What condition(s) must be satisfied so that the fraction of the energy radiated per period of the motion is small (*i.e.*, so that the quality factor of this oscillator remains high), and hence the trajectory is indeed nearly circular? Recall that the quantity $r_0 = e^2/mc^2$, where c is the speed of light, can be identified as the classical charge radius of the electron (in Gaussian units).

Verify that this requirement implies that the radiation-reaction force is small compared to the spring force on the electron.

It is not obvious that a classical system can be expected to satisfy these desirable requirements at all times. Nature avoids this difficulty by being quantum mechanical.

Supposing the motion of the charge to be nonrelativistic, show that the behavior of a quantum oscillator is such that the classical conditions for adiabatic damping are always satisfied.

Discuss also the case of relativistic motion. The essential quantum behavior here is contained in the insights of Hawking [1] and Unruh [2], the quantum "vacuum" appears to an accelerated observer to act like a thermal bath of temperature T related by $kT = \hbar a^\star/2\pi c$, where $a^\star$ is the acceleration in the (instantaneous) rest frame of the particle, $\hbar$ is Planck's constant, and k is Boltzmann's constant. For what radius of the electron's orbit does this relation imply that the thermal bath can in effect supply the energy radiated by the electron, such that the radius shrinks no further (and a laboratory observer considers the electron to have ceased radiating)?

2 Solution

2.1 Nonrelativistic Analysis

So long as the velocity v of the electron is small compared to c, the radiation is primarily dipole radiation, whose rate is given by the Larmor formula:

$$\frac{dU}{dt} = -\frac{2}{3}\frac{e^2 a^2}{c^3}\,, \tag{1}$$

where a is the acceleration of the particle.

For motion in a circle of radius r in a harmonic potential of spring constant K, Newton tells us that $F = ma = Kr$. Hence,

$$a = \frac{Kr}{m} \tag{2}$$

so that

$$\frac{dU}{dt} = -\frac{2}{3}\frac{e^2 K^2 r^2}{m^2 c^3}\,. \tag{3}$$

The energy U consists of the potential energy $Kr^2/2$ plus the kinetic energy $mv^2/2 = (mr/2)(v^2/r) = mra/2 = Kr^2/2$, so that

$$U = Kr^2 \tag{4}$$

(which is consistent with the virial theorem for bounded motion in a central potential). Thus, the differential equation for the time dependence of the energy is

$$\frac{dU}{dt} = -\frac{2}{3}\frac{e^2 K}{m^2 c^3}U = -\frac{2}{3}\frac{e^2}{mc^3}\frac{K}{m}U = -\frac{2}{3c}r_0\omega^2 U, \tag{5}$$

where $r_0 = e^2/mc^2$ is the so-called classical electromagnetic radius of the electron, and $\omega = \sqrt{K/m}$ is the angular frequency of the motion.

The solution to this equation is exponential decay of the energy, $U = U_0 e^{-t/\tau}$, where the time constant τ is

$$\tau = \frac{3c}{2r_0\omega^2}\,. \tag{6}$$

The condition for adiabatic decay with near-circular motion at all times is that τ be large compared to the period $2\pi/\omega$. Thus, we desire

$$\frac{c}{r_0\omega^2} \gg \frac{1}{\omega}\,, \qquad \text{and so} \qquad \omega \ll \frac{c}{r_0}\,. \tag{7}$$

It is more insightful to multiply this by the radius r of the electron's orbit, and to note that the velocity is $v = \omega r$, so the condition becomes

$$\frac{v}{c} \ll \frac{r}{r_0}\,. \tag{8}$$

This is fine so long as $r > r_0$, but we seem to be heading towards the conclusion that a classical analysis is doubtful for oscillations whose amplitude is less than the classical electron radius.

This impression is reinforced by consideration of the (Abraham-Lorentz) radiation reaction force,[1] which we recall to be $\mathbf{F}_{\text{reaction}} = (2/3c^3)e^2 d\mathbf{a}/dt$. It seems reasonable to expect that the reaction force be smaller than the drive force that causes it, which is the spring force Kr in the present problem.

[1]The radiation reaction force can be deduced from the relation between force and power, $\mathbf{F}_{\text{reaction}} \cdot \mathbf{v} = dU/dt$. In an example like the present case involving (nearly) uniform circular motion, the reaction force opposes the (nearly constant) velocity, and we can write $F_{\text{reaction}} = -(1/v)dU/dt = 2e^2 a\omega/3c^3$, noting that $a/v = v^2/rv = v/r = \omega$.

Now, for uniform circular motion with angular velocity $\boldsymbol{\omega}$ the rate of change of kinematic vectors such as $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$ is $d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r}$, $d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v}$ and $d\mathbf{a}/dt = \boldsymbol{\omega} \times \mathbf{a}$. In particular, the magnitude of the rate of change of the acceleration is $|d\mathbf{a}/dt| = \omega a$, so the magnitude of the radiation reaction force is

$$F_{\rm reaction} = \frac{2}{3c^3} e^2 \omega a = \frac{2}{3c^3} e^2 \omega \frac{Kr}{m} \,. \tag{9}$$

The condition that this force be small compared to the spring force Kr is

$$\frac{e^2}{mc^2} \frac{\omega}{c} = \frac{r_0 \omega}{c} \ll 1. \tag{10}$$

This is identical to the condition (7) found above that the motion be near circular at all times.

While we might not be troubled by classical motion that departs from the adiabatic condition of near-circular motion, a prediction that the radiation reaction force can exceed the drive force that causes it seems very odd.

It appears that Nature will never exhibit this oddity, if we take quantum behavior into account. In particular, the minimum energy quantum state of a three-dimensional harmonic oscillator is not zero, but rather $U_0 = 3\hbar\omega/2$. The wave function of this state has a finite extent, and is proportional to $e^{-r^2/2(\hbar/m\omega)}$. The characteristic spatial extent σ_r of this state is therefore

$$\sigma_r^2 = \frac{\hbar}{m\omega} = \left(\frac{\hbar}{mc}\right)^2 \frac{mc^2}{\hbar\omega} = \lambda\!\!\!^{-}_C{}^2 \frac{mc^2}{\hbar\omega} \,, \tag{11}$$

where $\lambda\!\!\!^{-}_C = \hbar/mc$ is the Compton wavelength of the electron.[2] So long as the quantum ground state of the oscillator has energy small compared to the rest energy of an electron (so that nonrelativistic quantum theory applies), we find that

$$\sigma_r > \lambda\!\!\!^{-}_C = \frac{\hbar}{mc} = \frac{e^2}{mc^2} \frac{\hbar c}{e^2} = \frac{r_0}{\alpha} \,, \tag{12}$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant. For elementary particles such as an electron or proton, $\alpha = 1/137$, and the spatial extent of the quantum ground state is large compare to the classical charge radius. Thus, quantum effects "protect" the amplitude of the particle's motion to have values larger than those for which the validity of a classical analysis becomes doubtful.

2.2 Relativistic Analysis

According to eq. (11), this quantum "protection" might not hold if $\hbar\omega \gg mc^2$, *i.e.*, if the oscillator is so strong that the motion of the charge is relativistic.

First, we consider the classical behavior of an electron in a very strong harmonic potential.

[2]Equation (11) for σ_r is, of course, consistent with the uncertainty principle that $\sigma_r^2 \sigma_p^2 \approx \hbar^2$, noting that $\sigma_p^2 \approx p^2 \approx m U_0 \approx \hbar m \omega$.

In the instantaneous rest frame of the electron, denoted by a $\star$, the Larmor formula (1) holds,

$$\frac{dU^\star}{dt^\star} = -\frac{2}{3}\frac{e^2 a^{\star 2}}{c^3}\,, \tag{13}$$

An unusual, but insightful, way of seeing this is to apply the equivalence principle to Hawking radiation [1], following Unruh [2], which tells us that the interaction of the accelerating particle with the quantum mechanical vacuum is as if the particle were immersed in a thermal bath of temperature T related by $kT = \hbar a/2\pi c$. We can consider the effective thermal bath seen by an accelerating electron to be a kind of radiation reaction not predicted in the classical theory.

The oscillating/accelerating electron cannot radiate away all of its energy because its interaction with the quantum vacuum causes it to absorb energy that continually re-excites the electron's oscillations. The thermal character of the Hawking-Unruh effect advises us that the particle will achieve a kind of thermal equilibrium when the effect (absorption of energy from the quantum vacuum) equals the cause (internal energy of the accelerating system), but the effect will never exceed the cause.

The condition of thermal equilibrium is, of course,

$$kT = U = Kr^2, \tag{14}$$

so together with the Hawking-Unruh relation we find

$$Kr^2 = \frac{\hbar a}{2\pi c} = \frac{\hbar K r}{2\pi m c}\,, \tag{15}$$

and so the equilibrium radius is given by

$$r = \frac{\hbar}{2\pi m c} = \frac{\lambda\!\!\!^{-}_C}{2\pi}\,. \tag{16}$$

Now the Compton wavelength $\lambda\!\!\!^{-}_C = \hbar/mc$ is related to the classical charge radius by $r_0 = \alpha \lambda\!\!\!^{-}_C$, where $\alpha = e^2/\hbar c$ is the fine-structure constant. For elementary particles such as an electron or proton, $\alpha = 1/137$ and the classical charge radius is small compared to the Compton wavelength. Thus, quantum effects "protect" the amplitude of the particle's motion to have values larger than those for which the validity of a classical analysis becomes doubtful.

The Hawking-Unruh argument finds a laboratory realization in electron storage rings (as used for synchrotron light sources), where a system of magnets provides a harmonic confining potential for motion transverse to the ring. The accelerated electrons do radiate, and the amplitude of their transverse oscillations does decrease with time. But this amplitude does not shrink to zero; rather it approaches a nonzero equilibrium value, the so-called quantum limit to synchrotron radiation damping. The technical details are slightly more complicated than in the present example, because the acceleration due to the circular motion of an electron in a storage ring is typically larger than its acceleration in the confining potential [3]

References

[1] S.W. Hawking, *Black hole explosions*, Nature **248**, 30-31 (1974),
http://kirkmcd.princeton.edu/examples/QED/hawking_nature_248_30_74.pdf
Particle Creation by Black Holes, Commun. Math. Phys. **43**, 199 (1975),
http://kirkmcd.princeton.edu/examples/QED/hawking_cmp_43_199_75.pdf

[2] W.G. Unruh, *Notes on black-hole evaporation*, Phys. Rev. D **14**, 870-892 (1976),
http://kirkmcd.princeton.edu/examples/QED/unruh_prd_14_870_76.pdf

[3] K.T. McDonald, *The Hawking-Unruh Temperature and Quantum Fluctuations in Particle Accelerators* (March 9, 1987),
http://kirkmcd.princeton.edu/accel/unruh.pdf

[4] K.T. McDonald, *The Hawking-Unruh Temperature and Damping in a Linear Focusing Channel* (Jan. 29, 1998), http://kirkmcd.princeton.edu/accel/linearchannel.pdf