Dielectric Cylinder Problem

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1 Problem

A cylinder of radius a and (relative) dielectric constant ϵ is placed along the z axis in an electric field whose form is $\mathbf{E}_i = E_0\hat{\mathbf{x}} + E_1[(x/a)\hat{\mathbf{x}} - (y/a)\hat{\mathbf{y}}]$ before the wire is placed in that field. The medium surrounding the wire is a liquid with relative dielectric constant $\epsilon' \neq 1$. The form of the initial electric field has been chosen so that there will be a force on the induced polarization due to the nonuniform field $\mathbf{E}_1 = E_1[(x/a)\hat{\mathbf{x}} - (y/a)\hat{\mathbf{y}}]$. You need NOT calculate that force, but do give expressions for the total electric field $\mathbf{E}$, the displacement field $\mathbf{D}$, and the polarization density $\mathbf{P}$ everywhere.

2 Solution

A solution to the extended version of this problem, cast in terms of magnetism rather than electricity, is at http://kirkmcd.princeton.edu/examples/permeable_wire.pdf

We adopt a coordinate system in which the axis of the wire is the z.

Because we are dealing with dielectric media with nonzero polarization $\mathbf{P}$, both the electric fields $\mathbf{E}$ and $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}$ are of utility. The initial external field is

$$\mathbf{E}_i = E_0\hat{\mathbf{x}} + E_1\left(\frac{x}{a}\hat{\mathbf{x}} - \frac{y}{a}\hat{\mathbf{y}}\right), \qquad \mathbf{D}_i = \epsilon'\mathbf{E}_i, \tag{1}$$

where ϵ' is the dielectric constant of the medium surrounding the cylinder. When the cylinder is placed into this medium, we expect a polarization force in the $+x$ direction due to the nonuniform field $\mathbf{E}_i$ that increases with x.

In addition to the rectangular coordinate system (x, y, z), we will work in a cylindrical coordinate system (r, θ, z). The usual transformation of the units vectors between these two coordinate systems are

$$\hat{\mathbf{x}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta}, \qquad \hat{\mathbf{y}} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\theta}, \tag{2}$$

and

$$\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}, \qquad \hat{\theta} = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}. \tag{3}$$

In static dielectric media the electric field $\mathbf{E}$ obeys $\nabla \times \mathbf{E}_{\rm ind} = 0$, so it may written as $\mathbf{E}_{\rm ind} = -\nabla\phi$ in terms of a scalar potential ϕ that obeys Laplace's equation, $\nabla^2\phi = 0$.

The external field (1) can be regarded as due to the scalar potential

$$\phi_i = -E_0 x - \frac{E_1}{2}\frac{x^2 - y^2}{a} = -E_0 r\cos\theta - \frac{E_1}{2}\frac{r^2}{a}\cos 2\theta. \tag{4}$$

The external field induces additional terms in the scalar potential that also vary as $\cos\theta$ or $\cos 2\theta$, since these are two of the set of orthogonal functions in which the scalar potential $\phi(r,\theta)$ can be expanded. In particular, we can write

$$\phi = \begin{cases} -E_0 r\cos\theta - \frac{E_1}{2}\frac{r^2}{a}\cos 2\theta + A_0\frac{r}{a}\cos\theta + \frac{A_1}{2}\frac{r^2}{a^2}\cos 2\theta & (r < a), \\ -E_0 r\cos\theta - \frac{E_1}{2}\frac{r^2}{a}\cos 2\theta + A_0\frac{a}{r}\cos\theta + \frac{A_1}{2}\frac{a^2}{r^2}\cos 2\theta & (r > a), \end{cases} \tag{5}$$

which is continuous at $r = a$. The fields obey the additional matching condition that the radial component $D_r = \epsilon E_r$ of the displacement field is continuous at $r = a$ (since $\nabla\cdot\mathbf{D} = 0$). As we have different dielectric constants ϵ for $r < a$ and ϵ' for $r > a$, the condition is that

$$\epsilon'\frac{\partial\phi(r = a^+)}{\partial r} = \epsilon\frac{\partial\phi(r = a^-)}{\partial r}, \tag{6}$$

and hence,

$$\begin{aligned} &\epsilon'\left(-E_0\cos\theta - E_1\cos 2\theta - \frac{A_0}{a}\cos\theta - \frac{A_1}{a}\cos 2\theta\right) \\ &= \epsilon\left(-E_0\cos\theta - E_1\cos 2\theta + \frac{A_0}{a}\cos\theta + \frac{A_1}{a}\cos 2\theta\right). \end{aligned} \tag{7}$$

The equality holds separately for the coefficients of the orthogonal functions $\cos\theta$ and $\cos 2\theta$, so that

$$A_{0,1} = \frac{\epsilon - \epsilon'}{\epsilon + \epsilon'}aH_{0,1}, \tag{8}$$

$$\phi = \begin{cases} -\frac{2\epsilon'}{\epsilon+\epsilon'}\left(E_0 r\cos\theta + \frac{E_1}{2}\frac{r^2}{a}\cos 2\theta\right) & (r < a), \\ -E_0\left(r - \frac{\epsilon-\epsilon'}{\epsilon+\epsilon'}\frac{a^2}{r}\right)\cos\theta - \frac{E_1}{2}\left(\frac{r^2}{a} - \frac{\epsilon-\epsilon'}{\epsilon+\epsilon'}\frac{a^3}{r^2}\right)\cos 2\theta & (r > a), \end{cases} \tag{9}$$

and

$$\begin{aligned} \mathbf{E} &= -\frac{\partial\phi}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} \\ &= \begin{cases} \frac{2\epsilon'}{\epsilon+\epsilon'}\left(E_0\cos\theta + E_1\frac{r}{a}\cos 2\theta\right)\hat{\mathbf{r}} - \frac{2\epsilon'}{\epsilon+\epsilon'}\left(E_0\sin\theta + E_1\frac{r}{a}\sin 2\theta\right)\hat{\theta} & (r < a), \\ \left[E_0\left(1 + \frac{\epsilon-\epsilon'}{\epsilon+\epsilon'}\frac{a^2}{r^2}\right)\cos\theta + E_1\left(\frac{r}{a} + \frac{\epsilon-\epsilon'}{\epsilon+\epsilon'}\frac{a^3}{r^3}\right)\cos 2\theta\right]\hat{\mathbf{r}} & \\ \quad - \left[E_0\left(1 - \frac{\epsilon-\epsilon'}{\epsilon+\epsilon'}\frac{a^2}{r^2}\right)\sin\theta + E_1\left(\frac{r}{a} - \frac{\epsilon-\epsilon'}{\epsilon+\epsilon'}\frac{a^3}{r^3}\right)\sin 2\theta\right]\hat{\theta} & (r > a), \end{cases} \\ &= \begin{cases} \frac{2\epsilon'}{\epsilon+\epsilon'}\left(E_0 + E_1\frac{x}{a}\right)\hat{\mathbf{x}} - \frac{2\epsilon'}{\epsilon+\epsilon'}E_1\frac{y}{a}\,\hat{\mathbf{y}} = \frac{2\epsilon'}{\epsilon+\epsilon'}\mathbf{E}_i & (r < a), \\ \left[E_0\left(1 + \frac{\epsilon-\epsilon'}{\epsilon+\epsilon'}\frac{a^2}{r^2}\cos 2\theta\right) + E_1\left(\frac{x}{a} + \frac{\epsilon-\epsilon'}{\epsilon+\epsilon'}\frac{a^3}{r^3}\cos 3\theta\right)\right]\hat{\mathbf{x}} & \\ \quad + \left[\frac{\epsilon-\epsilon'}{\epsilon+\epsilon'}E_0\frac{a^2}{r^2}\sin 2\theta + E_1\left(-\frac{y}{a} + \frac{\epsilon-\epsilon'}{\epsilon+\epsilon'}\frac{a^3}{r^3}\sin 3\theta\right)\right]\hat{\mathbf{y}} & (r > a). \end{cases} \end{aligned} \tag{10}$$

Of course,

$$\mathbf{D} = \begin{cases} \epsilon\mathbf{E} & (r < a), \\ \epsilon'\mathbf{E} & (r > a). \end{cases} \tag{11}$$

These forms obey the matching conditions that D_r and E_θ are continuous at the boundary $r = a$. Similarly, polarization is given by

$$\mathbf{P} = \begin{cases} \frac{\epsilon - 1}{4\pi}\mathbf{E} & (r < a), \\ \frac{\epsilon' - 1}{4\pi}\mathbf{E} & (r > a). \end{cases} \tag{12}$$