The Momentum of Heat

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(January 13, 2025)

When heat flows from a hotter body to a colder one, energy is transferred from the former to the latter. According to Einstein, this implies that the mass of the hotter body is reduced while that of the colder body increases. This transfer of mass in time between different regions of space suggests that some momentum is involved.

Already in 1900, Poincaré [1] argued that the flux of energy in the electromagnetic field, described by the Poynting vector $\mathbf{S}$, is associated with a density of momentum $\mathbf{S}/c^2$ in the electromagnetic field.

This type of relation is more general, as first discussed by Planck (1908, p. 829 of [2]), who argued that a flow $\mathbf{q}$ (with dimensions of energy per unit area per unit time) of heat is associated with a momentum density $\mathbf{p} = \mathbf{q}/c^2$, where c is the speed of light in vacuum.[1] The factor $1/c^2$ suggests that this very small momentum density could be called "relativistic",[2] as well as "hidden".

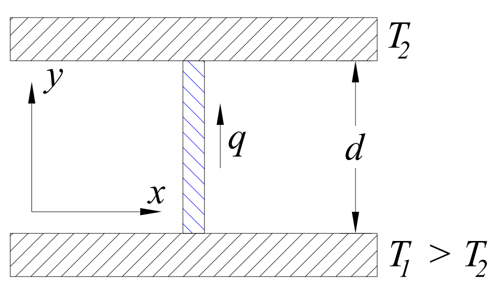
The concept of "hidden" momentum has been reviewed by the author in [7], where the "hidden" momentum of a subsystem is advocated to be

$$\mathbf{P}_{\rm hidden} \equiv \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area} = -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\, d{\rm Vol}, \qquad (1)$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total "mass", U is its total energy, $\mathbf{x}_{\rm cm}$ is its center of mass/energy, $\mathbf{v}_{\rm cm} = d\mathbf{x}_{\rm cm}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its "mass" density, u is its energy density, $\mathbf{v}_b$ is the velocity (field) of its boundary, and,

$$f^\mu = \frac{\partial \mathsf{T}^{\mu\nu}}{\partial x_\nu} = \partial_0 \mathsf{T}^{\mu 0} + \partial_j \mathsf{T}^{\mu j}, \qquad (2)$$

is the 4-force density exerted on the subsystem by the rest of the system, with $\mathsf{T}^{\mu\nu}$ being the stress-energy-momentum 4-tensor of the subsystem, and $x_\nu = (ct, \mathbf{x})$.



One type of heat flow is via radiant (electromagnetic) energy, which involves transfer of photons, that we readily associate with momentum. Here, we consider a case (also discussed in [9]) where radiant heat transfer is assumed to be negligible. Two heat reservoirs (with

[1]This type of relation was discussed by Eckart on p. 923 of [4], endorsed by Feynman in Sec. 27.6 of [5], and attributed to Planck [2] by Møller in eq. (13) of [6].

[2]The density $u_{\rm thermal}$ of thermal energy makes a contribution to the mass of $m_{\rm thermal} = u_{\rm thermal}/c^2$.

large but not infinite internal energy) at temperatures $T_1 > T_2$ are connected by a bar that supports heat flow $\mathbf{q} = q\,\hat{\mathbf{y}}$, which we approximate as uniform inside the bar, as sketched on the previous page. The heat reservoirs are so large that the rate q of the heat flow can be approximated as constant in time.

We now consider whether the subsystem of the bar contains "hidden" momentum, beginning our discussion in the rest frame of the system (in which the bar is at rest).

Since we ignore radiation, the stress-energy momentum 4-tensor[3] is purely "mechanical",

$$\mathsf{T}^{\mu\nu} = \left(\begin{array}{c|c} u & c\,\mathbf{p} \\ \hline c\,\mathbf{p} & -T^{ij} \end{array}\right), \tag{3}$$

where indices μ and ν take on values 0, 1, 2, 3, spatial indices i and j take on values 1, 2, 3, $\rho = u/c^2$ is the mass density of the bar, and T^{ij} is the "mechanical 3-stress tensor.

For a solid bar that is constrained by the heat reservoirs to have a constant length d in y, but with no constraints on the bar in x or z, the only nonzero component of the mechanical stress tensor is $\mathsf{T}^{yy} = \sigma = E\alpha(T_1 - T_2)$, where E is the Young's modulus and α is the thermal expansion coefficient.[4] Then, all components of $\mathsf{T}^{\mu\nu}$ are constant, and the 4-force density inside the bar is zero,

$$f^\mu = (0, 0, 0, 0). \tag{4}$$

and there is no "hidden" momentum in the bar (in its rest frame) according to the second form of eq. (1), since $f^0 = 0$.

We can also consider an inertial frame (the $'$ frame) in which the bar has nonzero velocity $\mathbf{v}$ with respect to the rest frame (for which the boost from the rest frame has velocity $-\mathbf{v}$). But, since the 4-force density is zero in the rest frame, it is also zero in any other inertial frame,

$$f'^{\,0} = (0, 0, 0, 0), \tag{5}$$

and according to eq. (1) there is no "hidden" momentum in any inertial frame.

This result disagrees with a claim in [9] that the "hidden" mommentum is nonzero in an inertial frame in which the bar has velocity $v\,\hat{\mathbf{x}}$.[5] It seems that the 4-force density was not carefully computed in that paper.

1 Discussion

If this system is closed and isolated, with no radiative heat transfer, its total momentum is constant, and there exists an inertial frame in which the total momentum is zero and the center of mass of the system is at rest. However, this frame is not the instantaneous rest frame of the system, as in the latter the center of mass of the system moves (very slowly) in the $+y$-direction.

[3] See, for example, Secs. 32-33 of [8].

[4] See, for example https://en.wikipedia.org/wiki/Thermal_stress

[5] Other comments by the present author related to the example of [9] are at [10].

As noted above, there exists a momentum density $\mathbf{q}/c^2$ associated with heat flux flux $\mathbf{q}$, and we can assign a velocity to the flux according to

$$\mathbf{v}_{\text{heat flow}} = \frac{\mathbf{q}}{u_{\text{heat}}}\,, \tag{6}$$

where u_{heat} is the density of thermal (heat) energy.[6] In the present example, the temperature inside the bar varies in y, so we assign a nonzero velocity (and momentum) to the heat flow at most points inside the bar in any inertial frame.

If we accept the claim that a heat flow $\mathbf{q}$ is associated with a momentum density $\mathbf{p} = \mathbf{q}/c^2$ in the rest frame of the bar, and also that we may neglect radiative heat transfer in the present example, then it seems that we know the stress-energy-momentum tensor (3) of the bar in its (inertial) rest frame. We then consider that there is no "hidden" momentum in this frame. And, as we can transform the stress-energy momentum tensor into any other inertial frame, it we also consider that there is no "hidden" momentum in any such frame.

The stress-energy-momentum tensor T' in a frame where the system has velocity $\mathbf{v} = v\,\hat{\mathbf{y}}$ with respect to its rest frame is related by $\mathsf{T}' = \mathsf{L}_y \mathsf{T} \mathsf{L}_y$ where the (tensor) Lorentz transformation L_y is

$$\mathsf{L}_y^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma & 0 & \gamma\beta & 0 \\ \hline 0 & 1 & 0 & 0 \\ \gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \approx \left(\begin{array}{c|ccc} 1 & 0 & \beta & 0 \\ \hline 0 & 1 & 0 & 0 \\ \beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \tag{7}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$, and the approximation holds for $v \ll c$, which suffices here. We find

$$\mathsf{T}\mathsf{L}_y \approx \left(\begin{array}{c|ccc} u & 0 & q/c & 0 \\ \hline 0 & 0 & 0 & 0 \\ q/c & 0 & \sigma & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c|ccc} 1 & 0 & \beta & 0 \\ \hline 0 & 1 & 0 & 0 \\ \beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|ccc} u + \beta q/c & 0 & q/c + \beta u & 0 \\ \hline 0 & 0 & 0 & 0 \\ q/c + \beta\sigma & 0 & \sigma + \beta q/c & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{8}$$

$$\mathsf{T}' = \mathsf{L}_y \mathsf{T} \mathsf{L}_y \approx \left(\begin{array}{c|ccc} u + 2\beta q/c & 0 & q/c + \beta(u+\sigma) & 0 \\ \hline 0 & 0 & 0 & 0 \\ q/c + \beta(u+\sigma) & 0 & \sigma + 2\beta q/c & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{9}$$

[6] The idea that an energy flux vector is the product of energy density and energy flow velocity seems to be due to Umov [11], based on Euler's continuity equation [12] for mass flow, $\boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = -\partial\rho/\partial t$.
For an electromagnetic example, see Sec. 2.1.4 of [13].

The momentum density inside the bar in the ′ frame is

$$\mathbf{p}' = \frac{\mathsf{T}'^{0y}}{c}\,\hat{\mathbf{y}} \approx \left(\frac{u'v}{c^2} + \frac{\sigma v}{c^2} + \frac{q}{c^2}\right)\hat{\mathbf{y}} = \left(\rho' v + \frac{\sigma v + q}{c^2}\right)\hat{\mathbf{y}} \approx \rho' v\,\hat{\mathbf{y}}, \tag{10}$$

where $u' = \mathsf{T}'^{00} = u + 2qv/c^2$ and $\rho' = u'/c^2$ is the mass density in ′ frame.

We could also suppose that the "bar" is a can of gas. If the gas had uniform pressure P, its "mechanical" 3-stress tensor would be $T^{ij} = -P\,\delta^{ij}$. In the present example, we might suppose the pressure is independent of x and z but varies linearly with y according to

$$P = P_1 - \frac{P_1 - P_2}{d} y \equiv P_1 - Ky \qquad \text{with} \qquad K = \frac{P_1 - P_2}{d} > 0, \tag{11}$$

for a bar of length d with pressure P_1 at its warmer end $y = 0$ and $P_2 < P_1$ at its cooler end $y = d$. If the "mechanical" stress tensor had the form $T^{ij} = -P\,\delta^{ij}$ with P as in eq. (11), then the 4-force density (2) would be

$$f^\mu = (0, 0, -K, 0), \tag{12}$$

which is nonzero throughout the bar. However, this does not make physical sense, as a bar at rest cannot have a nonzero internal force density.

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The equations on p. 6 before eqs. (4) state that the density $\mathbf{p}$ of momentum in the electromagnetic field is $\mathbf{S}/c^2$, where $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{B}$ (in Gaussian units).

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