The Linear Homopolar Generator

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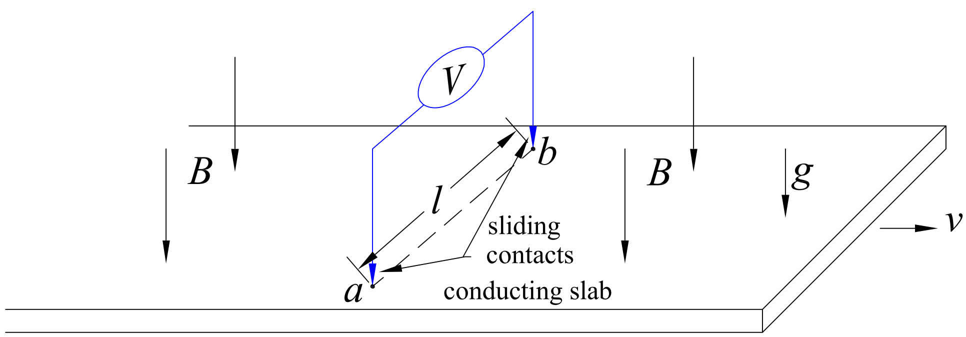
1 Problem

The linear homopolar generator is a variant of Faraday's disk dynamo that is perhaps of more pedagogic than practical interest. It is a kind of inverse of the "railgun" (linear homopolar motor) first considered by Ampère and de La Rive (1822) [1] and discussed by Maxwell in Arts. 594-596 of his *Treatise* [2].[1]

As sketched below, a basic configuration of the linear homopolar generator includes a conducting slab that moves to the right with constant velocity $\mathbf{v}$ in the lab frame while in a uniform, time-independent magnetic field $\mathbf{B}$ perpendicular to the slab. A voltmeter, at rest in the lab frame, is part of a closed circuit in a vertical plane with leads that make sliding contact with the slab at points a and b, where the distance from a to b is l.



What is the voltage ($\mathcal{EMF}$) measured by the voltmeter? Consider also the cases where any/all of the slab, voltmeter, and source of the magnetic field move to the right with velocity $\mathbf{v}$, supposing that the magnetic field is always uniform in the vicinity of the voltmeter.

You may assume that $v \ll c$, where c is the speed of light in vacuum.

2 Solution

This example was posed as Exercise 86, p. 262 of [4], and discussed in Sec. 9-5, p. 148 of [5] and in Example 4, Chap. 11, p. 394 of [6].

2.1 $\mathcal{EMF}$ and the Flux Rule

We take the view that the "flux rule" is simply the integral form of Faraday's law (whose differential form is $\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial t$), with the help of Stokes' theorem,

$$\oint_{\text{closed loop}} \mathbf{E} \cdot d\mathbf{l} = \int \boldsymbol{\nabla} \times \mathbf{E} \cdot d\mathbf{Area} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{Area} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{Area} = -\frac{d\Phi_{\mathbf{B}}}{dt}\,, \quad (1)$$

[1] Discussion by the author of railguns is in [3].

where in the case of a moving loop, bringing the time derivative outside the area integral changes the partial derivative to a total derivative. This is a consequence of one of Maxwell's equations, which are compatible with special relativity. Indeed, use of both Maxwell's second and fourth equations for a moving, rigid circuit leads to inconsistencies when using Galilean relativity, which are resolved by the Lorentz transformation of the electromagnetic fields, as discussed in Appendix C of [7].

Maxwell described the first integral in eq. (1) as the "electromotive force" ($\mathcal{EMF}$) in the first and last sentences of Art. 598 of [2].[2] With this usage, it is common to write the "flux rule" as,

$$\mathcal{EMF} = \oint_{\text{closed loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_{\mathbf{B}}}{dt}\,. \tag{2}$$

In Art. 598 of [2], Maxwell started from the integral form of Faraday's law, that the (scalar) *electromotive force* $\mathcal{E}$ in a circuit is related to the rate of change of the magnetic flux through it by his eqs. (1)-(2) (of Art. 598),

$$\mathcal{EMF} = -\frac{d\Phi_{\mathbf{B}}}{dt} = -\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{Area} = -\frac{d}{dt}\oint \mathbf{A} \cdot d\mathbf{l} = -\oint \left(\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A}\right) \cdot d\mathbf{l}, \tag{3}$$

where $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and the last form, involving the convective derivative, holds for a circuit that moves with velocity $\mathbf{v}$ with respect to the lab frame.[3] In his discussion leading to eq. (3) of Art. 598, Maxwell argued for the equivalent of use of the vector-calculus identity,

$$\boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}) = (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{v} + \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) + \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{v}), \tag{4}$$

which implies for the present case,

$$(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} = -\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) + \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}) = -\mathbf{v} \times \mathbf{B} + \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}), \tag{5}$$

$$\mathcal{EMF} = \oint \left(\mathbf{v} \times \mathbf{B} - \frac{\partial \mathbf{A}}{\partial t}\right) \cdot d\mathbf{l}, \tag{6}$$

since $\oint \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}) \cdot d\mathbf{l} = 0$.

Maxwell noted that a term of the form $-\boldsymbol{\nabla}\Psi$, where Ψ is a scalar field such as the electric scalar potential, could be added to the integrand of our eq. (6), his eq. (4) of Art. 598 of [2], with no effect on the integral. In his eq. (10) of Art. 599, Maxwell described the integrand as the *electromotive force* $\mathfrak{E} = \mathbf{v} \times \mathbf{B} - \partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi$. This has the implication that if the point with velocity $\mathbf{v}$ were occupied by an electric charge q it would experience force $\mathbf{F} = q(\mathbf{v} \times \mathbf{B} + \mathbf{E})$, where $\mathbf{E} = -\partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi$. That is, Maxwell had, in effect, stated the "Lorentz" force law in of Art. 599 [2], though this largely went unrecognized at that time.

Our eq. (6) corresponds to Maxwell's eq. (4) of Art. 598 of [2]. It appears not to be gauge invariant, and as such could be called "nonphysical". However, Maxwell's eq. (10) of

[2]In the third edition of Maxwell's *Treatise*, edited by J.J. Thomson after Maxwell's death, the term "electromotive force" in Arts. 598-600 was changed to "electromotive intensity".

[3]In Maxwell's notation, $E = \mathcal{E}$, $p = \Phi_{\mathbf{B}}$, $(F, G, H) = \mathbf{A}$, $(F\,dx/ds + G\,dy/ds + H\,dz/ds)\,ds = \mathbf{A} \cdot d\mathbf{l}$, $(dx/dt, dy/dt, dz/dt) = \mathbf{v}$, and $(a, b, c) = \mathbf{B}$. Note that we interpret Maxwell's $(d/dt)(F, G, H)$ as $\partial\mathbf{A}/\partial t$.

Art. 599 of [2] is gauge invariant, and better written with $\mathbf{E}$ rather than $-\partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi$. Then, we understand that Maxwell deduced an alternative form of our eq. (2),[4]

$$\mathcal{EMF} = \oint (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = \mathcal{EMF}_{\text{fixed loop}} + \mathcal{EMF}_{\text{motional}}, \tag{7}$$

where,

$$\mathcal{EMF}_{\text{fixed loop}}(t) = -\frac{\partial}{\partial t}\int_{\text{loop at time } t} \mathbf{B} \cdot d\mathbf{Area} = -\int_{\text{loop}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{Area} = \oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l}, \tag{8}$$

and,

$$\mathcal{EMF}_{\text{motional}} = \oint_{\text{loop}} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}, \tag{9}$$

in which $\mathbf{v}$ is the velocity (in the inertial lab frame of the calculation) of an element $d\mathbf{l}$ of the loop (which may or may not be conducting).

The two methods, eq. (2) and eq. (7), of computing induced $\mathcal{EMF}$s, give the same results (when correctly computed), although for examples with moving circuit elements, the method of eq. (7) is generally easier to apply.

The "flux rule" (2) is rather abstract for an arbitrary closed loop, and only has "practical" significance if the closed loop is an electrically conducting path. Even then, if the conductors along the loop are moving, the interpretation of eq. (2) is difficult,[5] such that many people, including Feynman [17], advocate use of our eqs. (7)-(9), with the view that the "flux rule" does not take motion of the loop into account, and is just our eq. (8),

$$\mathcal{EMF}_{\text{fixed loop}}(t) = -\int_{\text{loop at time, } t} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{Area}. \tag{10}$$

A review (with over 500 references) of the debate over the versions of the "flux rule" is given in [18], Sec. 2.4 of which includes six examples of circuits with moving parts that can be analyzed via our eq. (2) as well as our eqs. (7)-(9).[6]

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[4]The first clear statement of the equivalence of eqs. (2) and (7) may be in Sec. 86 of the text of Abraham (1904) [8], which credits Hertz (1890) [9] for inspiration on this. Boltzmann understood this equivalence in 1891 [10], but did not express it very clearly. An early verbal statement of this in the American literature was by Steinmetz (1908), pp. 1352-53 of [11], with a more mathematical version given by Bewley (1929) in Appendix I of [12]. Textbook discussions in English include that by Abraham Becker, pp. 139-142 of [4], by Sommerfeld, pp. 286-288 of [13], by Panofsky and Phillips, pp. 160-163 of [14], and by Zangwill, Sec. 14.4 of [15].

[5]An interesting qualification to the "flux rule" (2) was given by Carter (1967) on p. 170 of his "engineering" textbook [16]: *The equation $\mathcal{E} = -d\Phi/dt$ always gives the induced e.m.f. correctly, provided the flux-linkage is evaluated for a circuit so chosen that at no point are particles of the material moving across it.* That is, a valid path through the interior of a material must be at rest with respect to that material.

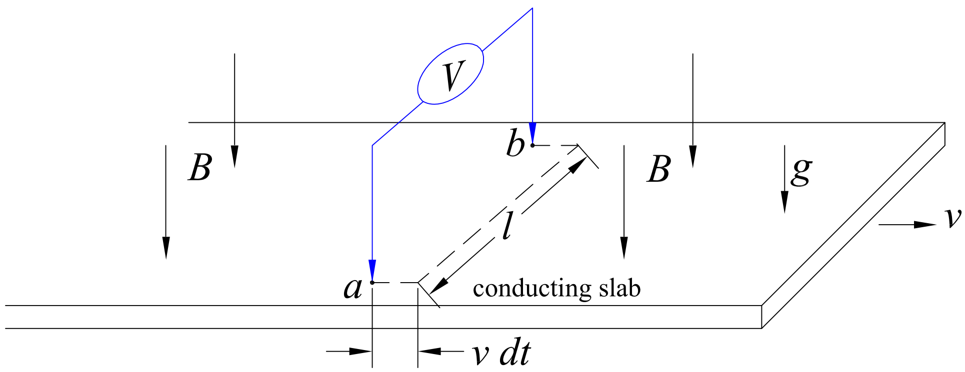
I believe this statement should also include the proviso: *..., and at no time is there a discontinuous change in the linked flux.*

Note that mention of *the material* implies the concept of $\mathcal{EMF}$ was only considered by Carter for conducting circuits, and not for "imaginary" closed curves. Further discussion of "Carter's Rule" is given in secs. 2.3-4 of [18].

[6]In circuit analyses, which rarely involve moving circuits, one speaks of the $\mathcal{EMF}$ of batteries and

2.2 Use of the Flux Rule for the Linear Homopolar Generator

For the basic case of a linear homopolar generator, sketched on p. 1 above, use of the flux rule (2) for a circuit that includes the dashed line from a to b would predict zero voltage ($\mathcal{EMF}$) to be measured by the voltmeter, in disagreement with experiment. However, the voltmeter circuit includes the moving slab, which should somehow be considered in the calculation. When current flows from sliding contact a to b it is not confined to any single line, so some representative line from a to b must be chosen. Following the advice of Carter mentioned in footnote 5 above, we consider the dashed line in the sketch below when evaluating the magnetic flux through the voltmeter circuit during a short time interval dt, which includes the line parallel to line ab at distance $v\,dt$ from it. This line moves along with the slab as time progresses, and no particle of the slab (that is at rest in the slab) crosses this line (or the two short, dashed line segments parallel to $\mathbf{v}$), as *per* Carter.



Then, during time dt the magnetic flux through the circuit is $d\Phi = Blv\,dt$ in magnitude, and the $\mathcal{EMF}$ measured by the voltmeter is

$$\mathcal{EMF} = -\frac{d\Phi}{dt} = -vBl. \tag{11}$$

2.3 Use of the Motional $\mathcal{EMF}$

For circuits with moving parts it is perhaps better to heed the advice of Feynman and others to use eqs. (7)-(9) for the circuit shown in the figure on p. 1, which includes the dashed line from a to b.

In this case, the $\mathcal{EMF}$ (8) through the fixed loop is zero, while the motional $\mathcal{EMF}$ (9) is

$$\mathcal{EMF}_{ab} = \int_a^b \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = vBl. \tag{12}$$

We will use the convention of [6] that the $\mathcal{EMF}$ in the voltmeter circuit is positive when it drives electric current in the direction from b to a. Then, the $\mathcal{EMF}$ in the voltmeter

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inductors (among other circuit elements), where this notion of $\mathcal{EMF}$ does not apply to a closed loop, but rather to a 2-terminal device. For terminals at points a and b, one often considers that $\mathcal{EMF}_{ab} = \int_a^b \mathbf{E} \cdot d\mathbf{l}$ along some path between a and b, typically defined by the circuit element of interest. But, in general, the integral depends on the path, such that this $\mathcal{EMF}$ is not well defined in a general mathematical sense. A review of this intricate issue is given in [19].

circuit is

$$\mathcal{EMF} = -\mathcal{EMF}_{ab} = -vBl, \tag{13}$$

in agreement with eq. (11).

2.4 Moving Voltmeter

The voltage measured by the voltmeter should always be computed in the rest frame of the voltmeter.

2.5 Moving Magnet[7]

We will consider variants of the linear homopolar generator in which the magnet that generates the uniform, time-independent magnetic field $\mathbf{B}$ has uniform velocity with respect to the voltmeter. In the frame of the voltmeter (which we will call the $'$ frame if this frame is not the lab frame) the magnetic field is still uniform and time independent, so (at least in the vicinity of the voltmeter) we have that $\boldsymbol{\nabla} \times \mathbf{E}' = -\partial \mathbf{B}'/\partial t = 0$. That is, the electric field due to the moving magnet in the frame of the voltmeter is electrostatic, and can be related to a scalar potential, so $\oint_C \mathbf{E}' \cdot d\mathbf{l}' = 0$, and the field $\mathbf{E}'$ induces no $\mathcal{EMF}$ around the closed loop containing the voltmeter.

For $v \ll c$ as assumed here, the magnetic field in the $'$ frame is $\mathbf{B}' = \mathbf{B}$, and the motional $\mathcal{EMF}$ due to this magnetic field is the same as if the magnet were not moving.

Hence, the total induced $\mathcal{EMF}$ (in the rest frame of the voltmeter) is the same whether of not the magnet is in motion.

2.6 8 Configurations of the Linear Homopolar Generator

We now survey the 8 possible variants of a linear homopolar generator considered in [5, 6].

2.6.1 $\mathbf{v}_{\rm slab} = 0$, $\mathbf{v}_{\rm magnet} = 0$, $\mathbf{v}_{\rm voltmeter} = 0$

In this trivial, static configuration no $\mathcal{EMF}$ is generated in the voltmeter circuit,

$$\mathcal{EMF}_1 = 0. \tag{14}$$

2.6.2 $\mathbf{v}_{\rm slab} = 0$, $\mathbf{v}_{\rm magnet} = \mathbf{v}$, $\mathbf{v}_{\rm voltmeter} = 0$

This case differs from the trivial case 2.6.1 only in that the magnet is moving, so according to Sec. 2.5, the $\mathcal{EMF}$ is that same as for that case,

$$\mathcal{EMF}_2 = 0. \tag{15}$$

[7]This argument follows Sec. 9-5 of [5].

2.6.3 $\mathbf{v}_{\rm slab} = \mathbf{v},\ \mathbf{v}_{\rm magnet} = \mathbf{v},\ \mathbf{v}_{\rm voltmeter} = \mathbf{v}$

As mentioned in Sec. 2.4 above, the computation of the $\mathcal{EMF}$ measured by the voltmeter should be done in the rest ($'$) frame of the voltmeter. In this frame both the slab and the magnet are at rest, so this is also a trivial, static case (like 2.6.1),

$$\mathcal{EMF}_3 = 0. \tag{16}$$

2.6.4 $\mathbf{v}_{\rm slab} = \mathbf{v},\ \mathbf{v}_{\rm magnet} = 0,\ \mathbf{v}_{\rm voltmeter} = \mathbf{v}$

In the rest ($'$) frame of the voltmeter the velocities are $\mathbf{v}'_{\rm slab} = 0$, $\mathbf{v}'_{\rm magnet} = -\mathbf{v}$ and $\mathbf{v}'_{\rm voltmeter} = 0$. This differs from the trivial case 2.6.2 only in the sign of the velocity of the magnet with respect to the $'$ frame of the voltmeter. Hence, we expect the $\mathcal{EMF}$ measured by the voltmeter to be zero,

$$\mathcal{EMF}_4 = 0. \tag{17}$$

2.6.5 $\mathbf{v}_{\rm slab} = \mathbf{v},\ \mathbf{v}_{\rm magnet} = 0,\ \mathbf{v}_{\rm voltmeter} = 0$

This is the basic case considered in Secs. 2.2 and 2.3 above,

$$\mathcal{EMF}_5 = -vBl. \tag{18}$$

2.6.6 $\mathbf{v}_{\rm slab} = \mathbf{v},\ \mathbf{v}_{\rm magnet} = \mathbf{v},\ \mathbf{v}_{\rm voltmeter} = 0$

This case differs from the basic case only by the velocity of the magnet, so according to Sec. 2.5 above the $\mathcal{EMF}$ measured by the voltmeter is the same as for the basic case,

$$\mathcal{EMF}_6 = -vBl. \tag{19}$$

2.6.7 $\mathbf{v}_{\rm slab} = 0,\ \mathbf{v}_{\rm magnet} = 0,\ \mathbf{v}_{\rm voltmeter} = \mathbf{v}$

In the $'$ (rest) frame of the voltmeter (where the measurement is made) the velocities are $\mathbf{v}'_{\rm slab} = -\mathbf{v}$, $\mathbf{v}'_{\rm magnet} = -\mathbf{v}$ and $\mathbf{v}_{\rm voltmeter} = 0$. We see that this case, in the $'$ frame, is formally equivalent to case 2.6.6 above, but with $\mathbf{v} \to -\mathbf{v}$. Hence, we infer that the $\mathcal{EMF}$ measured by the voltmeter is,

$$\mathcal{EMF}_7 = vBl. \tag{20}$$

2.6.8 $\mathbf{v}_{\rm slab} = 0,\ \mathbf{v}_{\rm magnet} = \mathbf{v},\ \mathbf{v}_{\rm voltmeter} = \mathbf{v}$

The case is the same as 2.6.7 but with a moving magnet, so according to Sec. 2.5 the $\mathcal{EMF}$ measured by the voltmeter is the same as for that case,

$$\mathcal{EMF}_8 = vBl. \tag{21}$$

These result agree with the tables on p. 149 of [5] (which gives only the magnitude of the $\mathcal{EMF}$) and p. 396 of [6].

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