The Electric Field
Outside a Rotating, Spherical Permanent Magnet

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(December 27, 2024)

1 Problem

What is the electric field outside an electrically neutral, sphere of uniform magnetization density $\mathbf{M}_0$ that rotates with constant angular velocity $\boldsymbol{\omega} \parallel \mathbf{M}_0$?

2 Solution

This solution follows [1, 2]. A slightly different analysis by the present author is at [3].

We consider a sphere of radius R, centered at the origin of the inertial (lab) frame, with uniform magnetization density $\mathbf{M}_0 = M_0\,\hat{\mathbf{z}}$ in its inertial rest frame, that rotates with constant angular velocity $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$ with respect to the lab frame. We also suppose that the surface velocity at the equator, $v = \omega R$ is small compared to the speed c of light in vacuum. The sphere has unit relative permittivity, $\epsilon = 1$ in Gaussian units (which will be used in this note), and the exterior of the sphere is vacuum.

A volume element of the sphere at distance ϱ from the z-axis has velocity $\mathbf{v} = \omega\varrho\,\hat{\boldsymbol{\phi}}$ in the lab frame, and in a cylindrical coordinate system (ϱ, ϕ, z). In the instantaneous inertial rest frame of this volume element, the magnetization density is $\mathbf{M}_0 = m_0\,\hat{\mathbf{z}}$, and the electric polarization density is $\mathbf{P}_0 = 0$.

The Lorentz transformation of the electric and magnetic polarization densities from the instantaneous inertial rest frame of the volume element to the lab frame is, for $v \ll c$,

$$\mathbf{P} = \gamma\left(\mathbf{P}_0 + \frac{\mathbf{v}}{c} \times \mathbf{M}_0\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{P}_0)\,\hat{\mathbf{v}} \approx \mathbf{P}_0 + \frac{\mathbf{v}}{c} \times \mathbf{M}_0 = \frac{\mathbf{v}}{c} \times \mathbf{M}_0 = \frac{\omega\varrho M_0}{c}\,\hat{\boldsymbol{\varrho}} \qquad (1)$$

$$\mathbf{M} = \gamma\left(\mathbf{M}_0 - \frac{\mathbf{v}}{c} \times \mathbf{P}_0\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{M}_0)\,\hat{\mathbf{v}} \approx \mathbf{M}_0 - \frac{\mathbf{v}}{c} \times \mathbf{P}_0 = \mathbf{M}_0. \qquad (2)$$

See, for example, [4]. In the lab frame the magnetization density $\mathbf{M}$ equals $\mathbf{M}_0$ and so is static. Hence the magnetic field $\mathbf{B}$ is as given in, for example, Sec. 5.10 of [5],

$$\mathbf{B}(r < R) = \frac{8\pi}{3}\mathbf{M} = \frac{8\pi M_0}{3}\,\hat{\mathbf{z}}, \qquad \mathbf{B}(r > R) = \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}}{r^3}\,, \qquad (3)$$

with $\mathbf{B}$ uniform inside the sphere, while outside the sphere $\mathbf{B}$ is equal to that of a "point" magnetic dipole $\mathbf{m} = 4\pi R^3 \mathbf{M}_0/3 = 4\pi R^3 M_0\,\hat{\mathbf{z}}/3$ at the origin, in a spherical coordinate system (r, θ, ϕ) based on the z-axis.

The electric-polarization density $\mathbf{P}$ in the lab frame is associated with a (constant) bound electric-charge density inside the sphere,

$$\rho_{\rm bound}(r < R) = -\boldsymbol{\nabla} \cdot \mathbf{P} = -\frac{1}{\varrho}\frac{\partial}{\partial\varrho}(\varrho P_\varrho) = -\frac{2\omega M_0}{c}\,, \qquad (4)$$

as well as with a bound surface-charge density

$$\sigma_{\text{bound}} = \mathbf{P} \cdot \hat{\mathbf{r}} = \frac{\omega \varrho M_0}{c} \hat{\boldsymbol{\varrho}} \cdot \hat{\mathbf{r}} = \frac{\omega \varrho M_0 \sin\theta}{c} = \frac{\omega M_0 R \sin^2\theta}{c}, \tag{5}$$

noting that $\varrho = R\sin\theta$ on the surface of the sphere.

The total bound electric charge inside the rotating, magnetized sphere is

$$Q_{\text{in}} = \frac{4\pi R^3}{3}\rho_{\text{bound}} = -\frac{8\pi R^3 \omega M_0}{3c}. \tag{6}$$

The total bound surface charge is

$$Q_{\text{surface}} = 2\pi R^2 \int_0^\pi \sin\theta\, d\theta\, \sigma_{\text{bound}} = \frac{2\pi R^3 \omega M_0}{c} \int_0^\pi \sin^3\theta\, d\theta = \frac{2\pi R^3 \omega M_0}{c} \left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^\pi = \frac{8\pi R^3 \omega M_0}{3c} = -Q_{\text{in}}, \tag{7}$$

using Dwight 430.30 [6]. The total bound electric charge is zero, as expected.[1]

The nonzero (static) electric field $\mathbf{E}$ in the lab frame associated with an electrically neutral, nonconducting, rotating magnetized sphere can be computed from the (static) electric scalar potential V and then computing $\mathbf{E} = -\boldsymbol{\nabla} V - \partial \mathbf{A} \partial ct$ (as the vector potential $\mathbf{A}$ of the static magnetization density $\mathbf{M}$ is independent of time, so $\partial \mathbf{A} \partial t = 0$.

We first note that the electric potential of a circular ring of uniformly distributed charge q that is at (R, θ', ϕ) in spherical coordinates can be expanded in a Legendre series as

$$V(r,\theta,\phi) = q \sum_{n=0}^\infty \frac{r_<^n}{r_>^{n+1}} P_n(\cos\theta') P_n(\cos\theta), \tag{8}$$

where $r_>$ ($r_<$) is the larger (smaller) of r and R. See, for example, Sec. 3-3 of [5].

In the present example, the bound surface charge (5) can be regarded as a series of rings of charge whose extent in θ' is $d\theta'$ with charge $dq = 2\pi R \sin\theta'\, d\theta'\, \sigma_{\text{bound}} = 2\pi R^3 \omega M_0 \sin^2\theta'\, d\cos\theta'/c$, such that the electric potential associated with the surface charge is

$$V_\sigma(r,\theta,\phi) = \frac{2\pi R^3 \omega M_0}{c} \sum_{n=0}^\infty \frac{r_<^n}{r_>^{n+1}} P_n(\cos\theta) \int_{-1}^1 \sin^2\theta' P_n(\cos\theta')\, d\cos\theta'. \tag{9}$$

With $P_0(x) = 1$ and $P_2(x) = (3x^2 - 1)/2 = (3x^2 - P_0)/2$, we have $x^2 = (P_0 + 2P_2)/3$, and

$$\sin^2\theta' = 1 - \cos^2\theta' = 1 - x^2 = P_0 - \frac{P_0 + 2P_2}{3} = \frac{2(P_0 - P_2)}{3}. \tag{10}$$

Hence,

$$V_\sigma(r,\theta,\phi) = \frac{4\pi R^3 \omega M_0}{3c} \sum_{n=0}^\infty \frac{r_<^n}{r_>^{n+1}} P_n(\cos\theta) \int_{-1}^1 [P_0(\cos\theta') - P_2(\cos\theta')] P_n(\cos\theta')\, d\cos\theta' = \frac{4\pi R^3 \omega M_0}{3c} \left(\frac{2}{r_>} P_0(\cos\theta) - \frac{2}{5}\frac{r_<^2}{r_>^3} P_2(\cos\theta)\right), \tag{11}$$

[1]This also follows from Gauss' law, $Q_{\text{in}} = \int \rho\, d\text{Vol} = -\int \boldsymbol{\nabla} \cdot \mathbf{P}\, d\text{Vol} = -\int \mathbf{P} \cdot d\mathbf{Area} = -\int \sigma\, d\text{Area} = -Q_{\text{surface}}$.

recalling that

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1}\delta_{mn}. \tag{12}$$

For $r > R$, $r_> = r$ and $r_< = R$, such that

$$V_\sigma(r > R, \theta, \phi) = \frac{4\pi R^3 \omega M_0}{3c}\left(\frac{2}{r} - \frac{2}{5}\frac{R^2}{r^3}\frac{3\cos^2\theta - 1}{2}\right), \tag{13}$$

Meanwhile, the electric scalar potential for $Q_{\rm in}$ and $r > R$ is simply $V_\rho(r > R) = Q_{\rm in}/r = -8\pi R^3 \omega M_0/3cr$, so the total electric potential for $r > R$ is

$$V(r > R) = V_\sigma(r > R) + V_\rho(r > R) = \frac{4\pi R^5 \omega M_0}{15c}\frac{1 - 3\cos^2\theta}{r^3}. \tag{14}$$

The electric field outside the rotating magnetized sphere has nonzero components

$$E_r(r > R) = -\frac{\partial}{\partial r}V(r > R) = \frac{4\pi R^5 \omega M_0}{5c}\frac{1 - 3\cos^2\theta}{r^4} = \frac{3B_{\rm in}}{10}\frac{\omega R}{c}\frac{R^4}{r^4}(1 - 3\cos^2\theta) \ll B_{\rm in}, \tag{15}$$

$$E_\theta(r > R) = -\frac{1}{r}\frac{\partial}{\partial\theta}V(r > R) = -\frac{4\pi R^5 \omega M_0}{5c}\frac{\sin 2\theta}{r^4}. \tag{16}$$

The (weak) exterior electric field falls off as $1/r^4$, and is a quadrupole field. The exterior electric field at the equator ($\theta = \pi/2$) points outwards from the origin, while the exterior electric field along the z-axis points inwards to the origin.

The nonzero exterior electric-field components in cylindrical coodinates are

$$E_\varrho(r > R) = E_r \sin\theta + E_\theta \cos\theta = \frac{4\pi R^5 \omega M_0}{5cr^4}\sin\theta(1 - 5\cos^2\theta) = \frac{4\pi R^5 \omega M_0}{5c}\frac{\varrho}{r^5}\left(1 - \frac{5z^2}{r^2}\right), \tag{17}$$

$$E_z(r > R) = E_r \cos\theta - E_\theta \sin\theta = \frac{4\pi R^5 \omega M_0}{5cr^4}(3\cos\theta - 5\cos^3\theta) = \frac{4\pi R^5 \omega M_0}{5c}\frac{z}{r^5}\left(3 - \frac{5z^2}{r^2}\right). \tag{18}$$

The transformation of the electromagnetic fields from the inertial lab frame to the rotating frame with angular velocity $\boldsymbol{\omega}$ with respect to the lab frame is, for $\omega\varrho \ll c$ (see, for example, eq. (43) of [7])

$$\mathbf{B}' = \mathbf{B}, \qquad \mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} = \mathbf{E} + \frac{\omega\varrho\,\hat{\boldsymbol{\phi}}}{c} \times \mathbf{B} \tag{19}$$

From eq. (3), the exterior magnetic field is, noting that $\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\boldsymbol{\varrho}}$.

$$\mathbf{B}'(r > R) = \frac{4\pi R^3 M_0}{3r^3}(3\cos\theta\,\hat{\mathbf{r}} - \hat{\mathbf{z}}) = \frac{4\pi R^3 M_0}{3r^3}[(3\cos^2\theta - 1)\,\hat{\mathbf{z}} + 3\cos\theta\sin\theta\,\hat{\boldsymbol{\varrho}}] = \frac{4\pi R^3 M_0}{3r^3}\left[\left(\frac{3z^2}{r^2} - 1\right)\hat{\mathbf{z}} + \frac{3\varrho z}{r^2}\,\hat{\boldsymbol{\varrho}}\right] \tag{20}$$

Then,

$$\frac{\omega\varrho\,\hat{\boldsymbol{\phi}}}{c}\times\mathbf{B}(r>R)=\frac{4\pi R^3\omega M_0\varrho}{3cr^3}\left[\left(\frac{3z^2}{r^2}-1\right)\hat{\boldsymbol{\varrho}}-\frac{3\varrho z}{r^2}\hat{\mathbf{z}}\right], \tag{21}$$

and the exterior electric field $\mathbf{E}'$ in the rotating frame has ϱ- and z-components

$$\begin{aligned}E'_\varrho(r>R)&=\frac{4\pi R^5\omega M_0}{5c}\frac{\varrho}{r^5}\left(1-\frac{5z^2}{r^2}\right)+\frac{4\pi R^3\omega M_0\varrho}{3cr^3}\left(\frac{3z^2}{r^2}-1\right)\\&=\frac{4\pi R^3\omega M_0\rho}{cr^3}\left(\frac{R^2}{5r^2}-\frac{R^2z^2}{r^4}+\frac{R^2}{3r^2}-\frac{1}{3}\right)=\frac{4\pi R^3\omega M_0\rho}{cr^3}\left(\frac{8R^2}{15r^2}-\frac{R^2z^2}{r^4}-\frac{1}{3}\right),\end{aligned} \tag{22}$$

$$\begin{aligned}E'_z(r>R)&=\frac{4\pi R^5\omega M_0}{5c}\frac{z}{r^5}\left(3-\frac{5z^2}{r^2}\right)-\frac{4\pi R^3\omega M_0\varrho^2z}{cr^5}\\&=\frac{4\pi R^3\omega M_0z}{cr^5}\left(\frac{3R^2}{5}-\frac{R^2z^2}{r^2}-\varrho^2\right).\end{aligned} \tag{23}$$

Most significantly, although the rotating frame is a rest frame for the magnetized sphere, the electric field in this frame is nonzero. Further, the electric field in the rotating frame is more complicated than in the lab frame, which is related to the "fictitious" charge distribution $\boldsymbol{\omega}\cdot\mathbf{H}/2\pi c$ that appears to exist according to observers in the rotating frame (see, for example, eq. (53) of [7]).

2.1 Rotating, Magnetized, Conducting Sphere

Most permanent magnets are conducting, so we also consider the case of a rotating, electrically neutral, magnetized conducting sphere.

The total force on conduction electrons inside the sphere must be zero, and the force on conduction electrons at the surface must be in the $\hat{\mathbf{r}}$ direction.

For steady motion, the velocity of a conduction electron of mass m and charge $-e$ at $\boldsymbol{\varrho}$ inside a rotating sphere as described above has velocity $\mathbf{v}=\boldsymbol{\omega}\times\boldsymbol{\varrho}$. The force on the electron is, ignoring gravity, that due to the Lorentz force and the centrifugal force

$$\mathbf{F}=-e\left(\mathbf{E}_{\rm in}+\frac{\boldsymbol{\omega}\times\boldsymbol{\varrho}}{c}\times\mathbf{B}_{\rm in}\right)+m\omega^2\boldsymbol{\varrho}=-e\left(\mathbf{E}_{\rm in}+\frac{8\pi\omega M_0}{3c}\boldsymbol{\varrho}\right)+m\omega^2\boldsymbol{\varrho}=0, \tag{24}$$

recalling eq. (3). Hence,

$$\mathbf{E}_{\rm in}=-\frac{8\pi\omega M_0}{3c}\boldsymbol{\varrho}+\frac{m\omega^2}{e}\boldsymbol{\varrho}. \tag{25}$$

The total charge density inside the rotating, conducting sphere is, noting that $\boldsymbol{\nabla}\cdot\boldsymbol{\varrho}=2$,

$$\rho_{\rm total}=\frac{1}{4\pi}\boldsymbol{\nabla}\cdot\mathbf{E}_{\rm in}=-\frac{4\omega M_0}{3c}+\frac{m\omega^2}{2\pi e}=\rho_{\rm bound}\left(\frac{2}{3}-\frac{m\omega\,c}{4\pi eM_0}\right)\equiv K\rho_{\rm bound}, \tag{26}$$

recalling eq. (4). As such, the total surface charge on the electrically neutral, rotating, conducting, magnetized sphere is $K\sigma_{\rm bound}$, and the exterior electric field is K times that of eqs. (15)-(16), where

$$K=\frac{2}{3}-\frac{m\omega\,c}{4\pi eM_0}. \tag{27}$$

Even for a permanent magnet with $B_{\rm in} \approx 1$ T, K differs from 2/3 by only $\approx 1\%$.

2.2 Rotating, Magnetized Cylinder

In case of an infinite, electrically neutral, nonconducting cylinder of radius R and symmetry axis z with uniform magnetization $M_0\,\hat{\mathbf{z}}$ in its inertial rest frame, when its steady rotation about the z-axis is $\omega\,\hat{\mathbf{z}}$ with respect to the inertial lab frame, its electric polarization density ρ is again given by eq. (1) and its bound electric charge density is again uniform as given by eq. (4). The bound charge per unit length along the cylinder is $\pi R^2 \rho = -2\pi R^2 \omega M_0/c$. Its bound surface charge density is $\sigma = \mathbf{P}\cdot\hat{\boldsymbol{\varrho}} = \omega R M_0/c$, with surface charge $Q_{\rm surface} = 2\pi R\sigma = 2\pi R^2 \omega M_0/c$ per unit length along the cylinder. The total (bound) charge per unit length along the rotating, magnetized cylinder is zero, and the exterior electric field is zero.

If the infinite, electrically neutral cylinder is conducting, there is a "correction" to the above values of the volume charge density ρ and the surface charge density σ, but the "correction" factor is the same for both. Hence, the exterior electric field is zero is this case also.

References

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