A Capacitor with Sliding Plates

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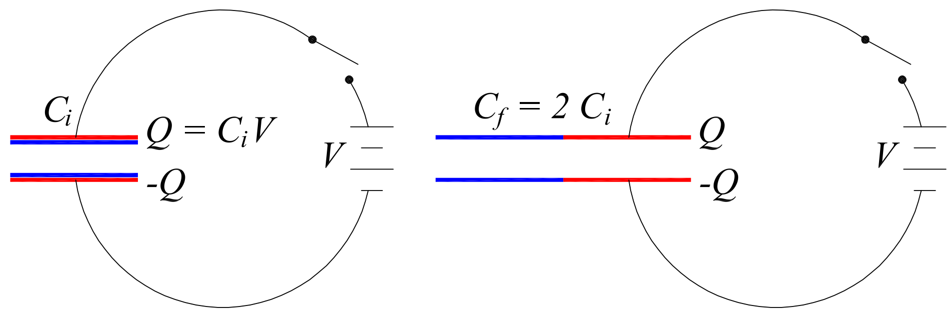
1 Problem

An isolated capacitor is charged with $\pm Q$ on its (rectangular) plates, which latter consist of two adjacent, identical layers that can slide with respect to one another. The plates are held apart at all times by a medium of uniform, relative dielectric constant, but the plates can slide without friction. Initially the four layers of the plates are aligned as in the left figure below, and the initial capacitance is C_i. Somehow the inner (blue) layers are (slowly) slid to the left until the configuration in the right figure below is achieved, with final capacitance $C_f = 2\,C_i$.

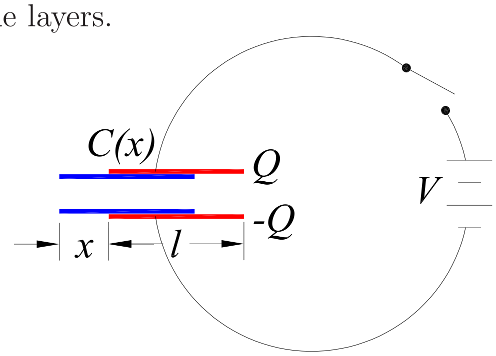


The initial electrostatic field energy of the system is $U_i = Q^2/2C_i$ (not counting the energy stored in the battery of voltage V), while the final field energy is $U_f = Q^2/2C_f = U_i/2$. What has happened to the "missing" energy $U_f - U_i/2 = U_i/2$ in the final state?

This problem was posed by Vladimir Onoochin.

2 Solution

Once the inner (blue) layers (of length l) have shifted by some amount x to the left as in the figure below, there is a nonzero electrical force $F(x)$ to the left on the blue layers. This force must be countered by an external force $-F(x)$ to the right on the blue layers to keep the velocity of the blue layers slow and constant, so this force does (negative) work as the blue layers move. That is, the "missing" energy is absorbed by the external agent the provides the needed force on the blue layers.



To be quantitative, we note that for slow, steady motion of the system, radiation can be ignored, and the electrical force on the blue layers is related by

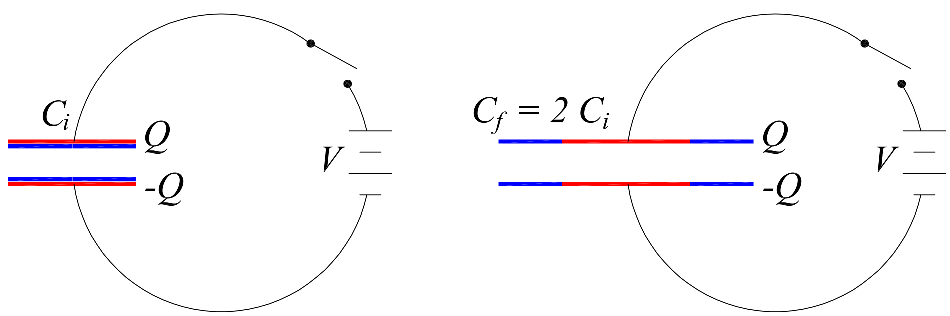
$$F(x) = -\frac{dU(x)}{dx}\,, \tag{1}$$

which points to the left. The external force is $F_{\rm ext} = -F(x)$, which points to the right, so the work done by the external agent is

$$W_{\rm ext} = \int_0^l -F_{\rm ext}\,dx = -\int_0^l \frac{dU(x)}{dx}\,dx = U_f - U_i = -\frac{U_i}{2}\,. \tag{2}$$

Thus, the energy of the external agent has increased by the "missing" energy $U_i/2$, as the blue layers slid by distance l to the left and the field energy dropped by $U_i/2$.[1]

The motion in the above scenario is not self starting, but requires an external perturbation to launch the blue layers to the left (plus an external force to hold the red layers at rest). A scenario that is self starting is sketched below, in which the blue layers are split in two equal halves, which repel one another once charged, and move to the two ends of the red layers.



Again (assuming that radiation can be neglected), an external agent absorbs the energy $U_i/2$ transfered from the initial electrostatic field energy $U_i = Q^2/2C_i$ into kinetic energy of the blue layers as they move from the initial to the final configuration.

This example is somewhat related to the more famous "two-capacitor" problem,
http://kirkmcd.princeton.edu/examples/twocaps.pdf

[1]In other scenarios, the external agent does not exert the full force $-F(x)$ on the sliding blue layers, but rather absorbs the nonzero kinetic energy of the blue layers when then arrive at $x = l$. In these scenarios, the accelerated, charged blue layers emit a small amount of electromagnetic radiation.