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# Deadtime When Using a FIFO Buffer

## 1 Introduction

This note concerns the calculation of deadtime in a data-acquisition system in which 'events' (here taken as a record of a fixed number of bits that is generated at some random time) are stored temporarily in a first-in/first-out buffer (FIFO) before being processed further. The average time between events is $t_1$. Whenever an event reaches the 'first' position in the FIFO processing begins on this event. After a processing time of $t_0$ the first event in the FIFO is cleared out and the next event (if any) becomes the first. The FIFO can store a maximum of $n$ events. Whenever the FIFO becomes full there is a a deadtime of $t_0$ before another event can be accepted. A general expression for the deadtime is somewhat complex, so we first consider limiting cases, then present a solution suitable for numerical evaluation, and finally give closed-form expressions for FIFO's with four or fewer event buffers.

## 2 Limiting Cases

Before analyzing the problem in detail we consider the limit of large and small values of $t_0/t_1$.

When the time between events $t_1$ is small compared to the processing time $t_0$, the deadtime approaches unity:

$$\lim_{t_0/t_1 \to \infty} D = 1.$$

When the time between events $t_1$ is large compared to the processing time $t_0$ the probability that another event occurs during the processing time of a previous event is small, and approaches $t_0/t_1$. The FIFO has a high probability of being empty at any time. An event will not be accepted only if all $n$ buffers in the FIFO are full. The probability for this is $(t_0/t_1)^n$, since we can ignore the small probability that the FIFO contained events at time $t_0$ before the arrival of the test event. However, when we discuss $n$ events occurring in some time interval, we mean an ordered sequence: the first event must come earlier than the second. Therefore we must divide the probability $(t_0/t_1)^n$ for any group of $n$ events during time $t_0$ by $n!$ to obtain the probability of an ordered sequence:

$$\lim_{t_0/t_1 \to 0} D = \frac{1}{n!} \left(\frac{t_0}{t_1}\right)^n .$$

This is, of course, the limiting result for the Poisson distribution,

$$\frac{(t_0/t_1)^n e^{-t_0/t_1}}{n!},$$

that $n$ events occur in time $t_0$.

# 3 General Analysis

The general analysis is more intricate because we cannot assume the FIFO is empty at any given time. We designate the probability that at some time the FIFO already contains $i$ events as $P_i$. For an $n$-deep FIFO we then have the normalization condition

$$\sum_{i=0}^{n} P_i = 1.$$

An event will be accepted so long as at least one buffer of the $n$-deep FIFO is free. Thus the deadtime is the probability that all $n$ buffers are full: $D = P_n$.

We assume the probabilities $P_i$ are independent of time.

We now relate the values of $P_i$ at time $t = 0$ to those at an earlier time. It seems simplest to chose that time to be $t = -t_0$, exactly one event-processing time earlier. Then in the absence of any new events in the interval $[-t_0, 0]$ the number of events in the FIFO will decrease by one.

We write

$$P_i(t=0) = \sum_{j=0}^{n} P_j(t=-t_0) Q_{i,j}, \qquad \text{or just} \qquad P_i = \sum_{j=0}^{n} P_j Q_{i,j},$$

where $Q_{i,j}$ is the probability that the FIFO makes a transition from containing $j$ to $i$ events during the interval $[-t_0, 0]$. We will find simple expressions for the $Q_{i,j}$ except for the $Q_{n,j}$ and for $Q_{n-1,n}$. The complexity in these cases is that during times when the FIFO is full any number of events can arrive without changing the state of the FIFO.

To have $i < n$ events in the FIFO at $t = 0$ starting from zero events in the FIFO at $t = -t_0$ we must have exactly $i$ events arriving in the interval $[-t_0, 0]$. Using the Poisson distribution we have

$$Q_{i,0} = \frac{(t_0/t_1)^i e^{-t_0/t_1}}{i!}, \qquad i < n.$$

If the FIFO already contains $j > 0$ events at $t = -t_0$ the first of these will be processed before $t = 0$ leaving $j - 1$ events in the FIFO. To end up with $i < n$ events in the FIFO at $t = 0$ exactly $i - (j - 1)$ events must arrive during the interval $[-t_0, 0]$. As a negative number events is impossible, the $Q_{i,j}$ will be nonzero only for $j \le i + 1$. The case $j = n$ is special because before the processing of the first event in the FIFO is completed any number of events could arrive without changing the state of the FIFO; these events would all be rejected. Hence $Q_{n-1,n}$ must be treated separately. The simple cases are

$$Q_{i,j} = 0, \qquad j > i + 1,$$

$$Q_{i,j} = \frac{(t_0/t_1)^{i-j+1} e^{-t_0/t_1}}{(i-j+1)!}, \qquad i < n,\ 0 < j \le i+1,\ \{i, j\} \ne \{n-1, n\}.$$

For $Q_{n-1,n}$, zero events can arrive after processing has finished on the first event in the FIFO. We suppose that this completion time is random within the interval $[-t_0, 0]$. Prior to that time any number of events can arrive (and be rejected) because the FIFO is still full. Hence

$$Q_{n-1,n} = \int_{-t_0}^{0} \frac{dt}{t_0} e^{-t/t_1} = \frac{t_1}{t_0}\left(1 - e^{-t_0/t_1}\right).$$

We do not need the $Q_{n,j}$ to complete the solution. We have expressions for the $P_i$ for $i < n$ so that $P_n$ can be found using the normalization that $\sum P_i = 1$.

A consistency check on the expression for $Q_{n-1,n}$ can be found by considering $Q_{n,n}$ (the simplest of the $Q_{n,j}$). Here we need exactly one event in the interval $[t, 0]$ after processing of the first event in the FIFO was finished at time $t$.

$$Q_{n,n} = \int_{-t_0}^{0} \frac{dt}{t_0} \frac{t}{t_1} e^{-t/t_1} = 1 - \frac{t_1}{t_0} \left(1 - e^{-t_0/t_1}\right).$$

We also note that $\sum_i Q_{i,j}$ is the probability of a transition to any number of events in the FIFO at $t = 0$ starting from $j$ events at $t = -t_0$. This probability is one:

$$\sum_{i=0}^{n} Q_{i,j} = 1.$$

For $j = n$ only $Q_{n-1,n}$ and $Q_{n,n}$ are non zero, so we must have $Q_{n-1,n} + Q_{n,n} = 1$, which is satisfied by our expressions.

We can now use the $n - 1$ relations

$$P_i = \sum_{j=0}^{i+1} Q_{i,j} P_j$$

(recalling that $Q_{i,j} = 0$ for $j > i + 1$) to deduce the constants $C_i$ defined by

$$P_i = C_i P_{i+1}.$$

Then we can also write

$$P_i = P_j \prod_{k=i}^{j-1} C_k, \qquad j > i.$$

The iterative procedure to find the $C_i$ uses

$$P_i = \sum_{j=0}^{i+1} Q_{i,j} P_j = Q_{i,i+1} P_{i+1} + P_i \sum_{j=0}^{i} Q_{i,j} \prod_{k=j}^{i-1} C_k,$$

where the ill-defined product

$$\prod_{k=i}^{i-1} C_k$$

is set to one. Then

$$C_i = \frac{Q_{i,i+1}}{1 - \sum_{j=0}^{i} Q_{i,j} \prod_{k=j}^{i-1} C_k}.$$

Now we can rewrite the normalization condition

$$\sum_{i=0}^{n} P_i = 1$$

as

$$1 = P_n \sum_{i=0}^{n} \prod_{j=i}^{n-1} C_j,$$

so the FIFO deadtime is

$$D = P_n = \frac{1}{\sum_{i=0}^{n} \prod_{j=i}^{n-1} C_j}.$$

For example, this procedure begins with

$$P_0 = Q_{0,0}P_0 + Q_{0,1}P_1,$$

which yields

$$P_0 = \frac{Q_{0,1}}{1 - Q_{0,0}} P_1 \equiv C_0 P_1.$$

Next

$$P_1 = Q_{1,0}P_0 + Q_{1,1}P_1 + Q_{1,2}P_2,$$

which yields

$$P_1 = \frac{Q_{1,2}}{1 - Q_{1,0}C_0 - Q_{1,1}} P_2 \equiv C_1 P_2,$$

*etc.*

Figure 1 shows the deadtime fraction $D$ calculated by the above prescription as a function of $a \equiv t_0/t_1$ from FIFO's with up to six event buffers. The Fortran program `fifo.for` for this can be found on the Princeton Technical Notes Web Page.

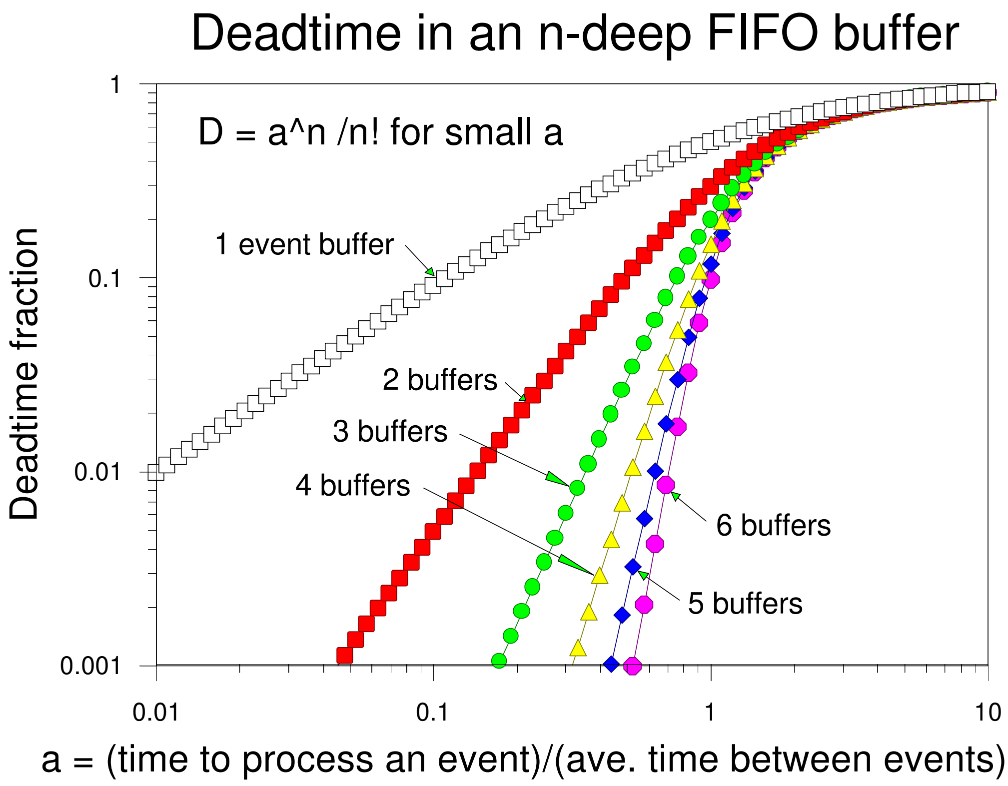


Figure 1: Deadtime fraction in an $n$-deep FIFO buffer.

# 4 Closed-Form Expressions

I have had the energy to carry out the above procedure analytically only up to $n = 4$. The forms below all display the limit that $D \to 1$ as $a \equiv t_0/t_1 \to \infty$. It is not hard to verify that $D \to a^n/n!$ for small $a$.

## 4.1 $n = 1$

$$D = P_1 = \frac{1}{1 + 1/a} = \frac{a}{1 + a}.$$

$$D(1) = 0.50.$$

## 4.2 $n = 2$

$$D = P_2 = \frac{e^a - 1 - a}{e^a - 1 - a + (e^a - 1)/a}.$$

$$D(1) = 0.30.$$

## 4.3 $n = 3$

$$D = P_3 = \frac{e^a - 1 - 2a - ae^{-a} + a^2 e^{-a}/2}{e^a - 1 - 2a - ae^{-a} + a^2 e^{-a}/2 + (e^a - 1)(1 - ae^{-a})/a}.$$

$$D(1) = 0.20$$

## 4.4 $n = 4$

$$D = P_4 = \frac{e^a - 1 - 3a + 2ae^{-a}(1 + a) - a^2 e^{-2a}(1/2 + a/6)}{e^a - 1 - 3a + 2ae^{-a}(1 + a) - a^2 e^{-2a}(1/2 + a/6) + (e^a - 1)(1 - 2ae^{-a} + a^2 e^{-2a}/2)/a}.$$

$$D(1) = 0.15.$$